Constant-Factor Approximation for ATSP with Two Edge Weights

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joint work with Ola Svensson and László A. Végh

EPFL

June 2, 2016

Traveling Salesman Problem

Given distances between n cities,
find the shortest tour which visits them all.

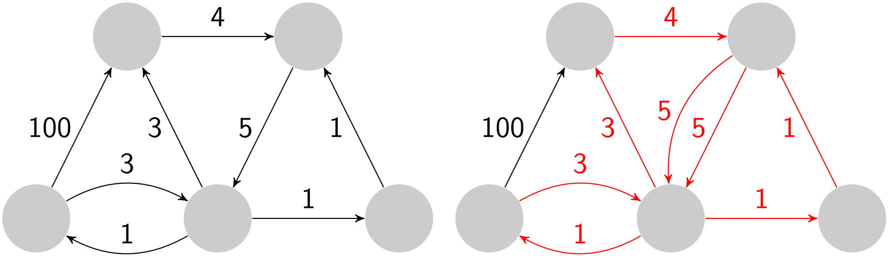
- Probably the best known NP-hard optimization problem
- Variants studied in mathematics as early as the 1800s
- Still huge gaps in understanding, especially of the asymmetric version

Definition of ATSP

Given: weighted directed graph $G = (V, E, w)$, $w : E \to \mathbb{R}_+$.

Find the cheapest multiset of edges $F \subseteq E$ such that the subgraph (V, F) is Eulerian and connected.

- Eulerian: for each vertex, indegree = outdegree.
- $w(F) = \sum_{e \in E} w(e)$: weight (cost) of tour.



Held-Karp relaxation

Write x_e for the number of times we traverse edge e and

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} w_e x_e \\ \text{subject to} \quad & x(\delta^+(v)) = x(\delta^-(v)) && \text{for all } v \in V, \\ & x(\delta^+(S)) \geq 1 && \text{for all } \emptyset \neq S \subsetneq V, \\ & x_e \geq 0 && \text{for all } e \in E \end{aligned}$$

where $\delta^+(v)$: outgoing edges of v, $\delta^-(v)$: incoming edges.

That is:

- x should be Eulerian,
- x should connect the entire graph.

Can be solved in polynomial time.

ATSP is NP-hard (even if G is unweighted, undirected etc.)

Main questions:

What is the best approximation ratio possible (in polynomial time)?

What is the integrality gap of the Held-Karp relaxation?

State of the art

- Approximation algorithms:
  - $\mathcal{O}\left(\frac{\log n}{\log \log n}\right)$-approximation algorithm [Asadpour et al. 2010]
  - lower bound: 75/74-approximation is NP-hard [Karpinski et al. 2013]
- Integrality gap:
  - upper bound: $\mathcal{O}(\mathrm{poly} \log \log n)$ [Anari, Oveis Gharan 2014]
  - lower bound: 2 [Charikar et al. 2006]
  - (smaller gap between lower and upper bounds)
- Is there an $\mathcal{O}(1)$-approximation algorithm?

Special cases

What if we assume something about G?

Oveis Gharan, Saberi 2011

$\mathcal{O}(1)$-approximation algorithm for ATSP on bounded-genus graphs (incl. planar graphs)

(because bounded-genus graphs have $\mathcal{O}(1)$-*thin trees*)

For symmetric TSP, since 2010, improvements when G is unweighted (graph TSP). What about ATSP?

Svensson 2015

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Our results

Svensson 2015

$\mathcal{O}(1)$-approximation algorithm for ATSP on unweighted graphs

- Implies $\mathcal{O}(w_{\max}/w_{\min})$-approximation in general – but this ratio can be unbounded
- Next logical step?

This work: Svensson, T., Vegh 2016

$\mathcal{O}(1)$-approximation algorithm for ATSP on graphs with two edge weights

(also a constant bound on the integrality gap)

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Local-Connectivity ATSP

Svensson 2015

$\mathcal{O}(1)$-approximation algorithm for ATSP on unweighted graphs

follows by:

- defining a new easier problem called Local-Connectivity ATSP
- reduction (technical core of paper):

For any class of graphs, if can approximate Local-Connectivity ATSP well, then can approximate ATSP well!

- can indeed approximate Local-Connectivity ATSP well for unweighted graphs (easy part of paper)

(note similarity with the thin tree approach)

For what other classes of graphs can we approximate Local-Connectivity ATSP well?

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More motivation

- A lot of work to prove the reduction
- But approximating Local-Connectivity ATSP on unweighted graphs is easy
- Now makes sense to put more work into the latter

Good sign: previously $\mathcal{O}(1)$-approximation for unweighted ATSP was unknown – now it follows easily

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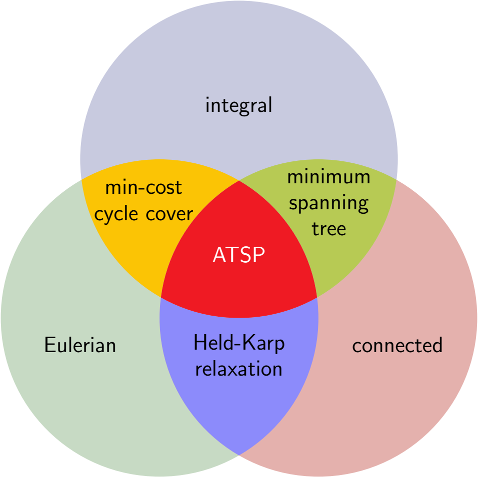
Pick two

Want the cheapest $x : E \to \mathbb{R}_+$ which touches every vertex and is:



Everything in the diagram is easy, except for ATSP!


JUST RELAX
CONNECTIVITY

Relaxing connectivity

Repeated cycle cover algorithm [Frieze et al. 1982]

- Pick cheapest cycle cover (**polytime solvable**)
- "Contract"
- Repeat

Number of phases = approximation ratio = $\log n$
because **connectivity is too weak**

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Number of phases = approximation ratio = log n
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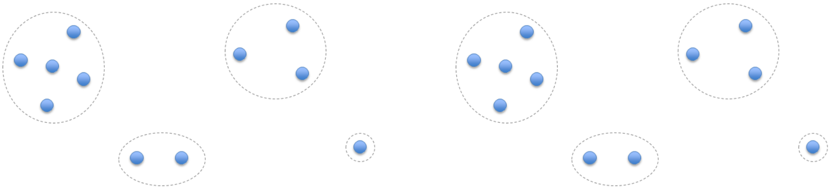
Find a subproblem which **yields stronger connectivity** than min-cost cycle cover
but **weaker** than ATSP
(maybe not polytime solvable)

Local-Connectivity ATSP

Given: weighted directed graph $G = (V, E, w)$, $w : E \to \mathbb{R}_+$,
and a partitioning $V = V_1 \cup \ldots \cup V_k$**.**

Find a **cheap** multiset of edges $F \subseteq E$ such that the subgraph (V, F) is Eulerian and **each cut** $(V_i, \overline{V_i})$ **is crossed**.



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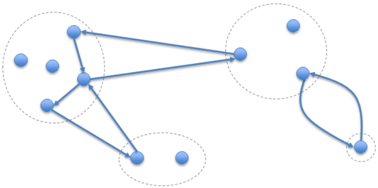

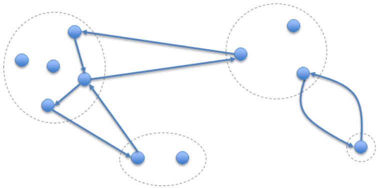

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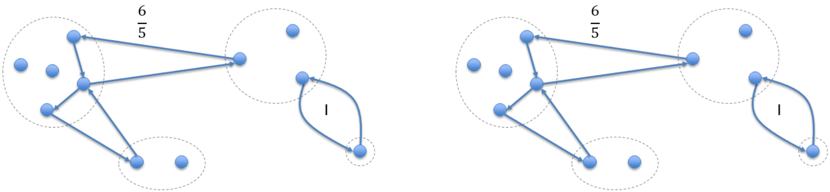
Algorithm is α-**light** if each component in (V, F) is **locally cheap**: $\frac{\#\text{ edges}}{\#\text{ vertices}} \leq \alpha$. *(oversimplified)*



Framework:

$G = (V, E, w) \quad \Longleftarrow$

$\Longrightarrow$ $\mathrm{lb} : V \to \mathbb{R}_+$ with $\sum_{v \in V} \mathrm{lb}(v) = \mathrm{OPT}_{\mathrm{LP}}$

partitioning $V = V_1 \cup \ldots \cup V_k \quad \Longleftarrow$

$\Longrightarrow$ solution F which is locally cheap w.r.t. lb

Local-Connectivity ATSP

An α-*light* algorithm for Local-Connectivity ATSP has two phases:

- Given $G = (V, E, w)$, output $\mathrm{lb} : V \to \mathbb{R}_+$ with $\sum_{v \in V} \mathrm{lb}(v) = \mathrm{OPT}_{\mathrm{LP}}$.
  - lb stands for "lower bound": a way to distribute the LP-lower-bound among vertices.

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- Now, given also a partitioning $V = V_1 \cup ... \cup V_k$, find multiset of edges $F \subseteq E$ such that:
  - subgraph (V, F) is Eulerian,
  - each V_i-cut is crossed: $|F \cap \delta^+(V_i)| \geq 1$ for $i = 1, ..., k$,
  - F is cheap, even *locally*: for each connected component of (V, F) we have $\frac{\text{weight of edges in component}}{\mathrm{lb}\text{ of vertices in component}} \leq \alpha$.

Local-Connectivity ATSP

An α-*light* algorithm for Local-Connectivity ATSP has two phases:
Compared to an α-approximation for ATSP (w.r.t. HK relaxation):

- Given $G = (V, E, w)$, output $\mathrm{lb} : V \to \mathbb{R}_+$ with $\sum_{v \in V} \mathrm{lb}(v) = \mathrm{OPT}_{\mathrm{LP}}$. (Not present in ATSP.)
  - lb stands for "lower bound": a way to distribute the LP-lower-bound among vertices.
- Now, given also a partitioning $V = V_1 \cup \ldots \cup V_k$, (not given) find multiset of edges $F \subseteq E$ such that:
  - subgraph (V, F) is Eulerian,
  - each V_i-cut is crossed: $|F \cap \delta^+(V_i)| \geq 1$ for $i = 1, \ldots, k$,
    In ATSP, every cut is crossed: $|F \cap \delta^+(S)| \geq 1$ for $\emptyset \subsetneq S \subsetneq V$,
  - F is cheap, even *locally*: for each connected component of (V, F) we have $\frac{\text{weight of edges in component}}{\mathrm{lb}\text{ of vertices in component}} \leq \alpha$.
    In ATSP, F is globally cheap: $\frac{w(F)}{\mathrm{lb}(V)} = \frac{w(F)}{\mathrm{OPT}_{\mathrm{LP}}} \leq \alpha$.

Local-Connectivity ATSP

Recap:

- the connectivity requirement is relaxed: rather than crossing all cuts, F only needs to cross each component of the partition $V = V_1 \cup ... \cup V_k$,
- the cost requirement is strengthened: rather than being cheap as a whole, F needs to be cheap locally at each connected component (w.r.t. some lb function).

Local-Connectivity ATSP

For any class of graphs:

Fact

If there is an α-approximation for ATSP (w.r.t. HK relaxation), then there is an α-light algorithm for Local-Connectivity ATSP.

Proof.

Output any $\mathrm{lb} : V \to \mathbb{R}_+$ such that $\sum_{v \in V} \mathrm{lb}(v) = \mathrm{OPT}_{\mathrm{LP}}$.
Disregard the partitioning and just run the ATSP algorithm.
If its output is globally cheap, it's also locally cheap
(since there is only one connected component). □

Theorem (Svensson)

If there is an α-light algorithm for Local-Connectivity ATSP, then:

- *the integrality gap of the Held-Karp relaxation is at most 5α,*
- *there is a 9.001α-approximation algorithm for ATSP.*

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And:

Fact

There is a 3-light algorithm for Local-Connectivity ATSP on unweighted graphs.

So:

Theorem

There is a 27-approximation algorithm for ATSP on unweighted graphs.

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If:

Fact

There is a $\mathcal{O}(1)$-light algorithm for Local-Connectivity ATSP on some class of graphs.

Then:

Theorem

There is a $\mathcal{O}(1)$-approximation algorithm for ATSP on that class of graphs.

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For any class of graphs:

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And:

Theorem (Svensson, T., Vegh [IPCO 2016])

There is a $\mathcal{O}(1)$-light algorithm for Local-Connectivity ATSP on graphs with two edge weights.

So:

Theorem

There is a $\mathcal{O}(1)$-approximation algorithm for ATSP on graphs with two edge weights.

How to solve Local-Connectivity ATSP?

As a warmup, we show:

Fact

There is a 3-light algorithm for Local-Connectivity ATSP on unweighted graphs.

Even simpler: assume the given partitioning is the **singleton partition** $V = \{v_1\} \cup ... \cup \{v_n\}$.

How to solve Local-Connectivity ATSP?

Recap of L-C ATSP for unweighted G, singleton partition

Given $G = (V, E)$ (unweighted), want:

- $\mathrm{lb} : V \to \mathbb{R}_+$ such that $\sum_{v \in V} \mathrm{lb}(v) = \mathrm{OPT}_{\mathrm{LP}}$,
- $F \subseteq E$: Eulerian multiset of edges

such that

- each singleton cut is crossed: $|F \cap \delta^+(v)| \geq 1$ for all $v \in V$,
- locally at each connected component $\widetilde{G}$ of (V, F), F is cheap: $|F \cap E(\widetilde{G})| \leq 3 \cdot \mathrm{lb}(\widetilde{G})$.

We round the LP solution $x^\star$:
Define lb so that each node "pays" for its outgoing edges:

$$\mathrm{lb}(v) := x^\star(\delta^+(v)) = \sum_{e \in \delta^+(v)} x^\star_e$$

And pick an integral solution $z = \mathbb{1}_F$ to the circulation problem:

$$1 \leq z(\delta^+(v)) \leq \lceil x^\star(\delta^+(v)) \rceil \qquad (z = x^\star \text{ is feasible})$$

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- locally at each connected component $\widetilde{G}$ of (V, F), F is cheap: $|F \cap E(\widetilde{G})| \le 3 \cdot \mathrm{lb}(\widetilde{G})$.

True for any $\widetilde{G} \subseteq V$:

$$|F \cap E(\widetilde{G})| \le \sum_{v\in\widetilde{G}} z(\delta^+(v)) \le \sum_{v\in\widetilde{G}} \lceil x^\star(\delta^+(v)) \rceil \le \sum_{v\in\widetilde{G}} 2x^\star(\delta^+(v)) = 2\mathrm{lb}(\widetilde{G}).$$

Crucial: rounding up is fine because $x^\star(\delta^+(v)) \ge 1$.

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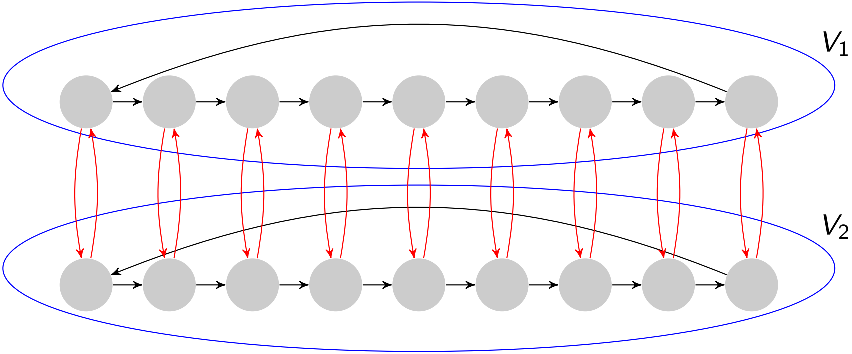
How to solve Local-Connectivity ATSP?

- Got 2-light algorithm
- Dealing with arbitrary partitions $V = V_1 \cup \ldots \cup V_k$ makes it 3-light

Two edge weights

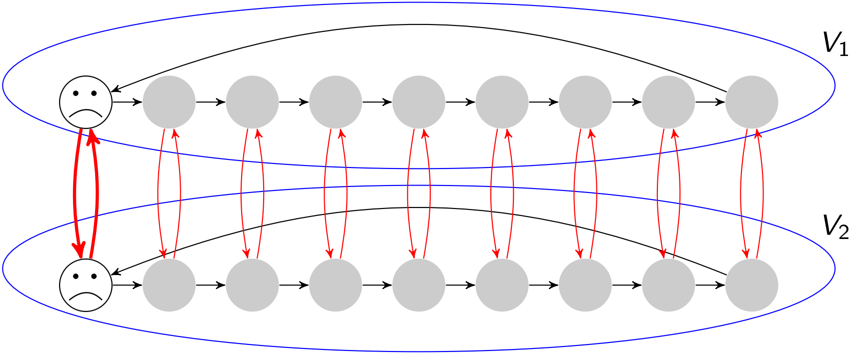
- Why not just do the same?



- Black edges are cheap and have $x_e^\star = 1 - \frac{1}{k}$
- Red edges are expensive and have $x_e^\star = \frac{1}{k}$
- In $x^\star$, each vertex pays only for $\frac{1}{k}$ expensive flow: the **thick red solution** can't be paid locally

Two edge weights

- Why not just do the same?



- Black edges are cheap and have $x_e^\star = 1 - \frac{1}{k}$
- Red edges are expensive and have $x_e^\star = \frac{1}{k}$
- In $x^\star$, each vertex pays only for $\frac{1}{k}$ expensive flow: the **thick red solution** can't be paid locally

Two edge weights

Problem: rounding red $x^\star$-flow from ε to 1 is too expensive

Solution: group small chunks of red $x^\star$-flow together and then round them

- we use a **flow theorem** to find a small set T of **terminals**
- we reroute red $x^\star$-flow to these terminals so that any path that uses an expensive edge must then go a terminal
- we put higher lb on terminals so the red $x^\star$-flow can be paid for there

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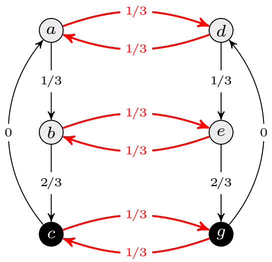
The flow theorem

Let E_1: expensive edges.

Theorem

There is a set of terminals $T \subseteq V$ and a flow f from the tails of expensive edges to T which:

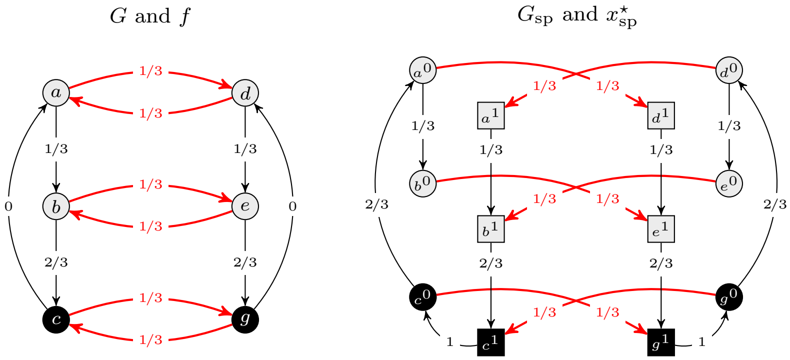
- $f \leq x^\star$
- *f saturates all expensive edges and has value $x^\star(E_1)$*
- *T is small: $|T| \leq 8x^\star(E_1)$*



The picture shows f.
Red edges are expensive.
$x^\star$ is $1/3$ for expensive edges
and $2/3$ for cheap (black) edges.
Terminals T are black.

Splitting the graph

We use f and T to split G and $x^\star$:



- copies v^1 carry the f-flow, copies v^0 carry the rest $(x^\star - f)$
- now any cycle with an expensive edge must visit a terminal

And pick an integral solution $z = \mathbb{1}_F$ to the circulation problem:

$$1 \leq z(\delta^+(v)) \leq \lceil 2x^\star_{\mathrm{sp}}(\delta^+(v)) \rceil$$

Splitting the graph

- This mostly does the trick for the singleton partitioning.
- More work needed in the general case.

Future work

- What wider classes of graphs admit an $\mathcal{O}(1)$-approximation?
  - Even the case of three edge weights is unsolved

- Beat $\mathcal{O}(\log n / \log\log n)$ for general case
  - Can we match the known integrality gap upper bound $\mathcal{O}(\mathrm{poly}\log\log n)$?

Thank you for your attention!