Matching is in QUASI-NC

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joint work with Ola Svensson

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

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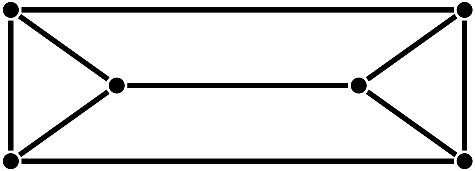
Perfect matching problem

Given a graph, can we pair up all vertices using edges?



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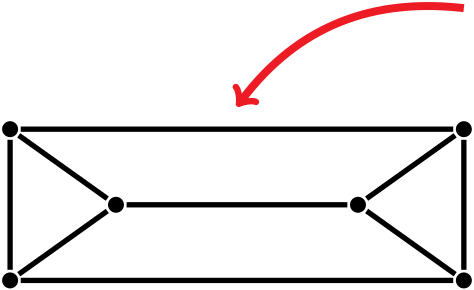
very tough instance: graph is non-bipartite!



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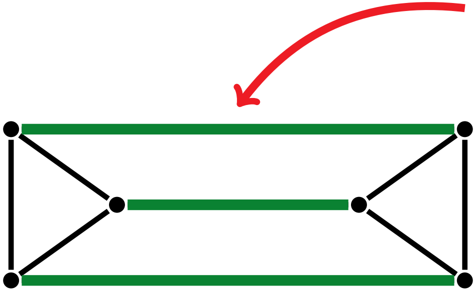
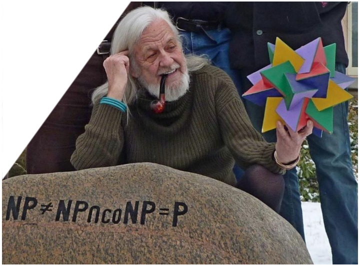


Perfect matching problem

Benchmark problem in computer science

Algorithms:

- bipartite: Jacobi [XIX century, weighted!]
- general: Edmonds [1965]
  - polynomial-time = efficient
- since then, tons of research
  and still active
- many models of computation:
  monotone circuits, extended formulations,
  parallel, distributed, streaming/sublinear, ...



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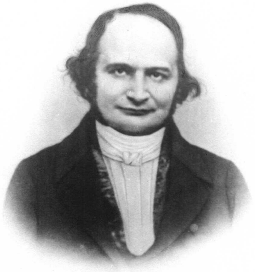

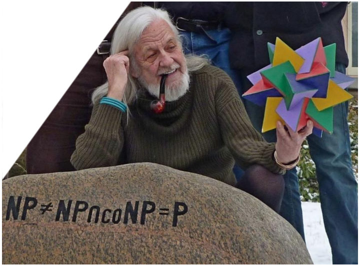

Parallel complexity

Class $\mathcal{NC}$: problems that paralellize completely

poly n processors

poly log n time

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Main open question: is matching in $\mathcal{NC}$?

Parallel complexity

Class $\mathcal{NC}$: problems that paralellize completely

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it's in RANDOMIZED $\mathcal{NC}$

poly log n time

Main open question: is matching in $\mathcal{NC}$?

Parallel complexity

- Matching is in RANDOMIZED $\mathcal{NC}$ [Lovász 1979]:
  has randomized algorithm that uses:
  
  - polynomially many processors
  - polylog time
- Search version is in RANDOMIZED $\mathcal{NC}$:
  
  - [Karp, Upfal, Wigderson 1986]
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first matching algorithm
to use Tutte's matrix
and Zippel-Schwartz Lemma

led to understanding of
computational relationship between
search and decision problems

introduced
the Isolation Lemma

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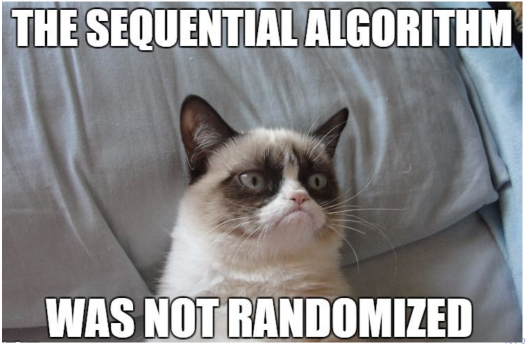

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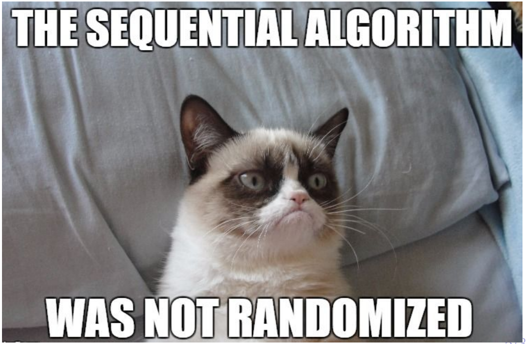
Can we derandomize
all efficient computation?



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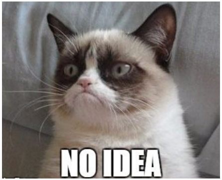

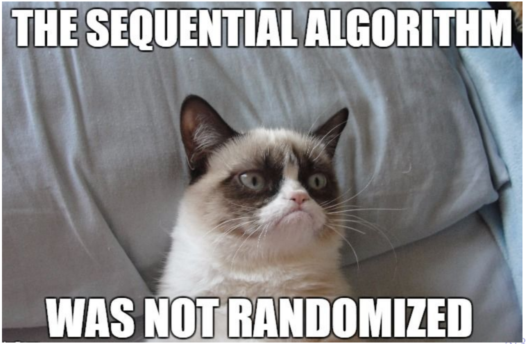

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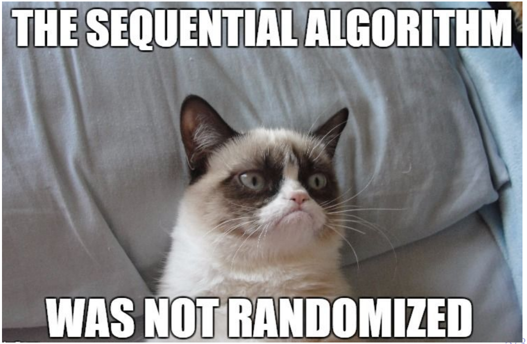

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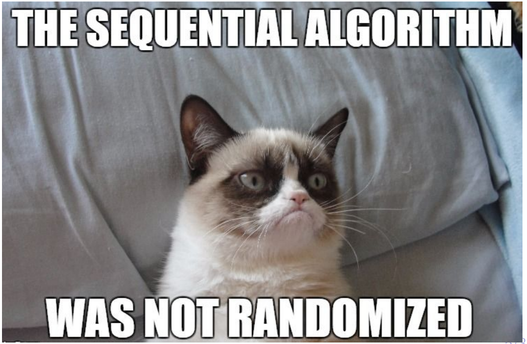

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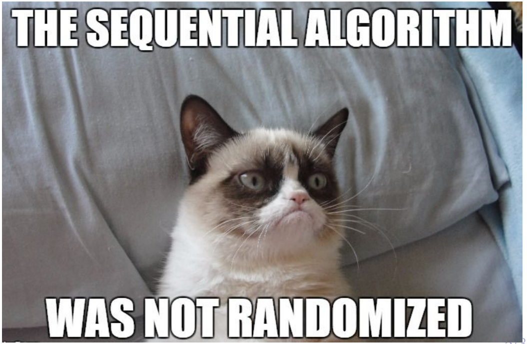

Is matching in $\mathcal{NC}$?

Yes, for restricted graph classes:

- bipartite regular [Lev, Pippenger, Valiant 1981]
- bipartite convex [Dekel, Sahni 1984]
- incomparability graphs [Kozen, Vazirani, Vazirani 1985]
- bipartite graphs with small number of perfect matchings [Grigoriev, Karpinski 1987]
- claw-free [Chrobak, Naor, Novick 1989]
- $K_{3,3}$-free (decision version) [Vazirani 1989]
- planar bipartite [Miller, Naor 1989]
- dense [Dahlhaus, Hajnal, Karpinski 1993]
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Is matching in $\mathcal{NC}$?

Fenner, Gurjar and Thierauf [2015] showed:

- Bipartite matching is in QUASI-$\mathcal{NC}$
  ($n^{\text{poly}\log n}$ processors, $\text{poly}\log n$ time, deterministic)

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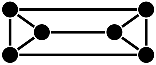
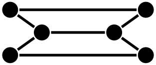
- Approach fails for non-bipartite graphs

 much harder than 

Our result

We show: general matching is in QUASI-$\mathcal{NC}$:

- $n^{\text{poly}\log n}$ processors
- $\text{poly}\log n$ time
- deterministic

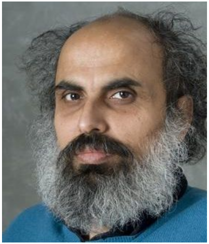
Outline

1. Isolating weight functions
   [Mulmuley, Vazirani, Vazirani 1987]

2. Bipartite case
   [Fenner, Gurjar, Thierauf 2015]

3. Difficulties of general case
   & our approach

1. Isolating weight functions [Mulmuley, Vazirani, Vazirani 1987]



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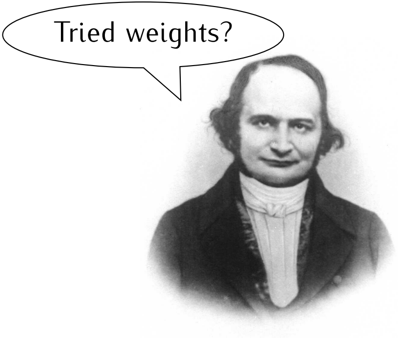

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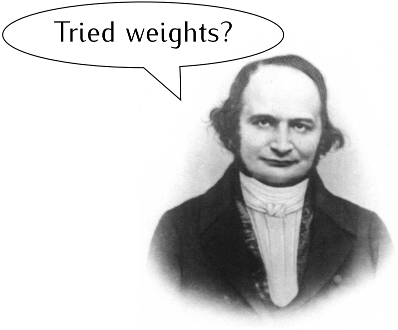

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Solution: look for a **min-weight** perfect matching

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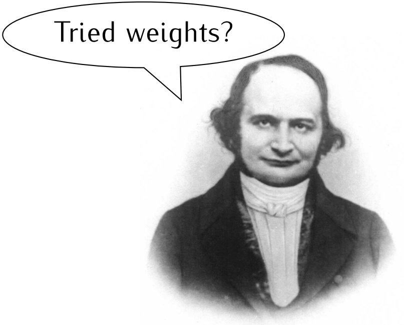

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MAKE LIFE HARDER

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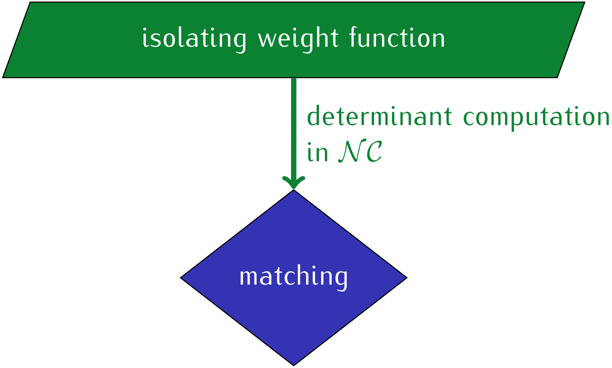
> Weight function $w : E \to \mathbb{Z}_+$ is **isolating**
> if there is a **unique** min-weight perfect matching

[Mulmuley, Vazirani, Vazirani 1987]



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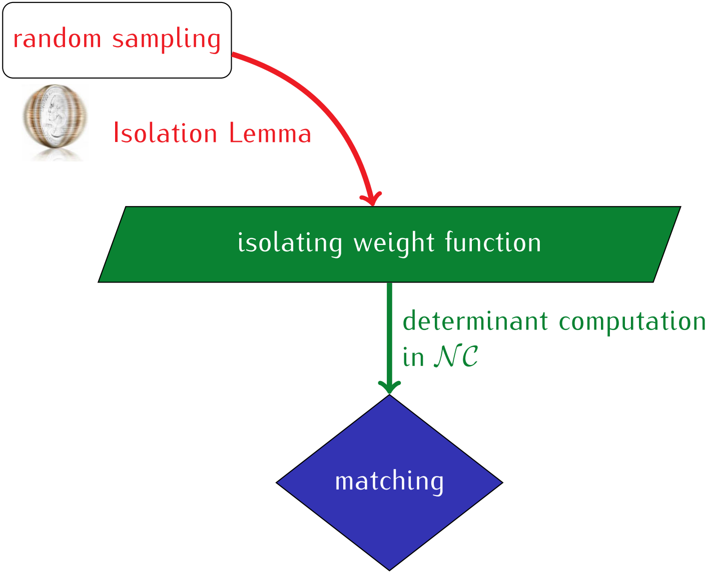

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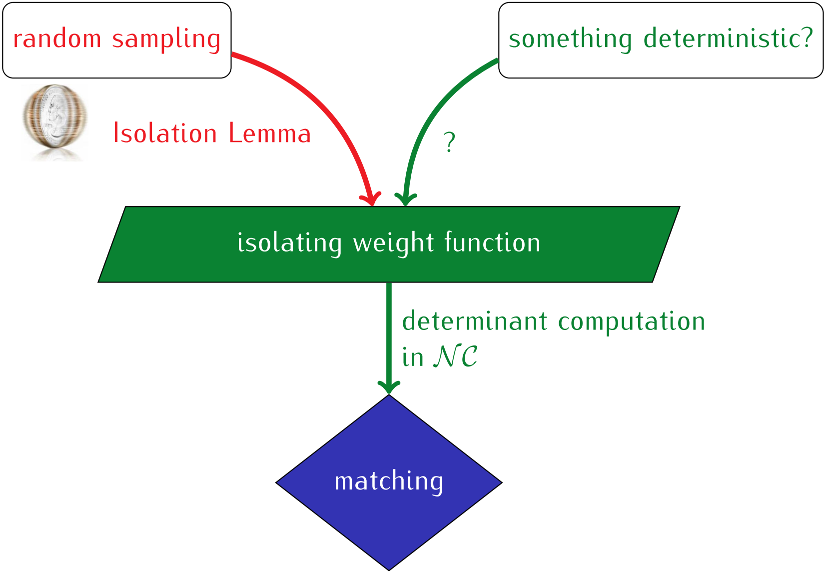

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Weight function $w : E \to \mathbb{Z}_+$ is **isolating**
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Given isolating w, can find perfect matching in $\mathcal{NC}$

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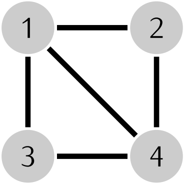

$$T(G) = \begin{pmatrix} 0 & X_{12} & X_{13} & X_{14} \\ -X_{12} & 0 & 0 & X_{24} \\ -X_{13} & 0 & 0 & X_{34} \\ -X_{14} & -X_{24} & -X_{34} & 0 \end{pmatrix}$$

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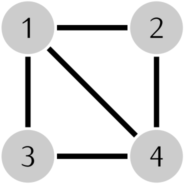

$$T^w(G) = \begin{pmatrix} 0 & 2^{w(1,2)} & 2^{w(1,3)} & 2^{w(1,4)} \\ -2^{w(1,2)} & 0 & 0 & 2^{w(2,4)} \\ -2^{w(1,3)} & 0 & 0 & 2^{w(3,4)} \\ -2^{w(1,4)} & -2^{w(2,4)} & -2^{w(3,4)} & 0 \end{pmatrix}$$

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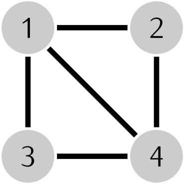

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- we can compute determinant in $\mathcal{NC}$

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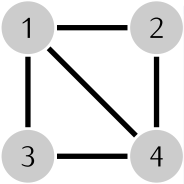

Why not just use

$$w(e_i) = 2^i?$$

It's clearly isolating...

3) $2^{w(1,4)}$
$2^{w(2,4)}$
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,4) 0

- build Tutte'
- det $T(G) \neq$ matching
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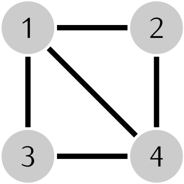
Given poly-bounded isolating w, can find perfect matching in $\mathcal{NC}$



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- we can compute determinant in $\mathcal{NC}$ **(if w poly-bounded)**

Isolation Lemma

Weight function $w : E \to \mathbb{Z}_+$ is **isolating**
if there is a **unique** min-weight perfect matching

Isolation Lemma [MVV 1987]

If each $w(e)$ picked randomly from $\{1, 2, \ldots, n^3\}$,
then $\mathrm{P}[w \text{ isolating}] \geq 1 - \frac{1}{n}$

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- holds more generally,
  for any set family in place of matchings!
- many applications in complexity theory
- related to Polynomial Identity Testing

Derandomize the Isolation Lemma

- **Challenge**:
  get an isolating weight function
  deterministically in $\mathcal{NC}$

- **We prove**:
  can construct $n^{O(\log^2 n)}$ weight functions in QUASI-$\mathcal{NC}$
  such that one of them is isolating

- We do it without looking at the graph

- **Implies: matching is in QUASI-$\mathcal{NC}$**

Special case of derandomizing Polynomial Identity Testing
– for the polynomial being $\det T(G)$

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Special case of derandomizing Polynomial Identity Testing
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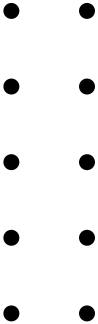
2. Bipartite case
[Fenner, Gurjar, Thierauf 2015]

Goal: how to construct $n^{O(\log n)}$ weight functions such that one of them is isolating?

Isolating matchings

What if w is **not** isolating?

- there are perfect matchings M, M' with $w(M) = w(M')$ minimum



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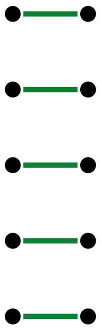

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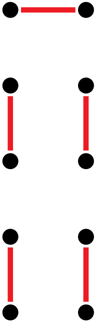

Isolating matchings

What if w is **not** isolating?

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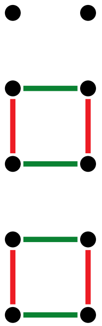
- symmetric difference $=$ alternating cycles



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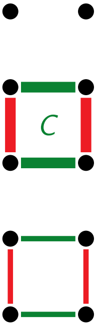

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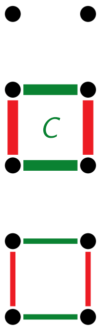

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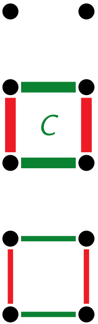

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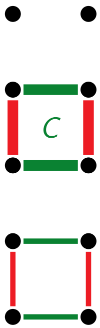

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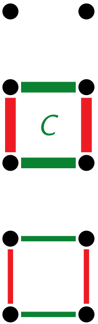

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New objective: assign $\neq 0$ discrepancy to every cycle

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There is a poly-sized set $\mathcal{W}$ of weight functions such that:
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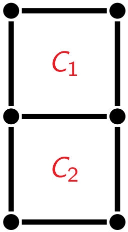
Not so easy, but we can cope with all **4-cycles**.

Removing cycles



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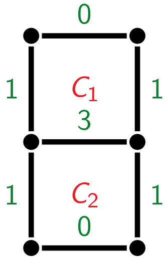

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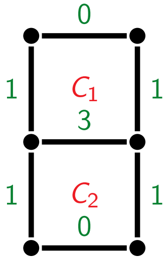

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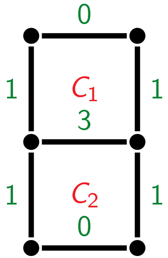
those edges that are in a min-weight perfect matching



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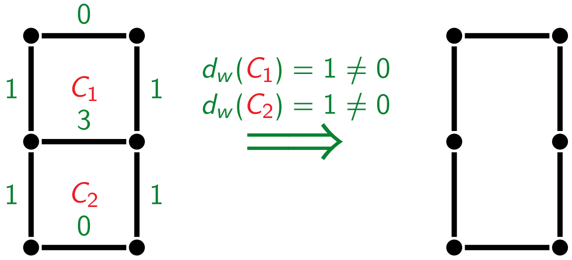

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Once we assign a cycle $\neq 0$ discrepancy,
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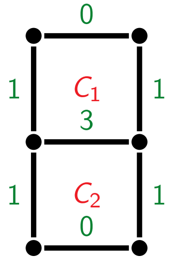

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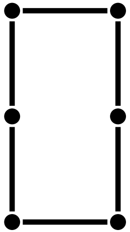

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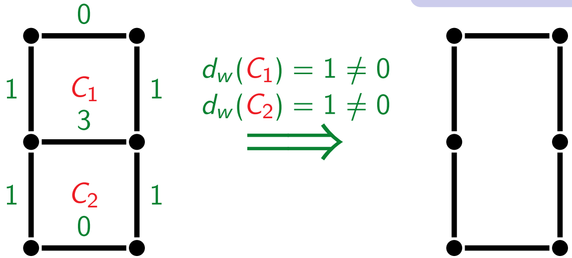

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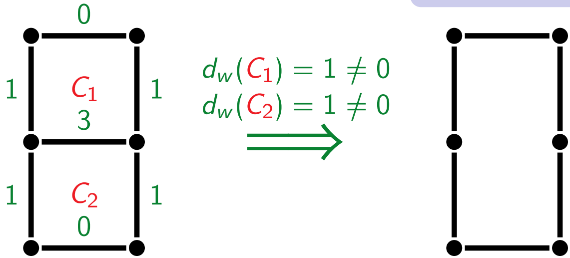

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By assigning $\neq 0$ discrepancy to 4-cycles, we can remove them.
Then continue restricted to the smaller active subgraph!

Proof of bipartite key property

Bipartite key property

Once we assign a cycle $\neq 0$ discrepancy,
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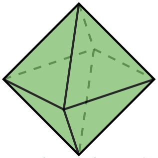
Proof:

- Let $\mathcal{M}$ be the set of perfect matchings minimizing w
- Consider the convex hull of $\mathcal{M}$ (face F of the bipartite matching polytope):



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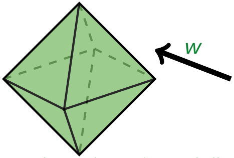

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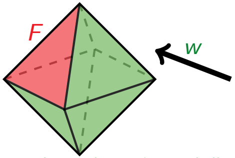

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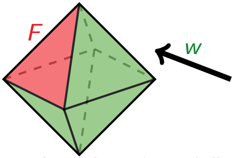

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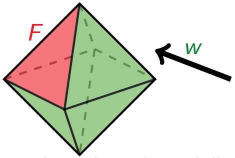

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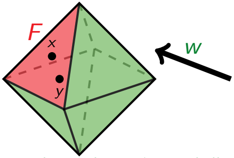

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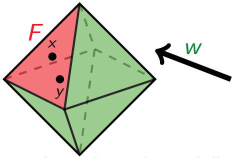

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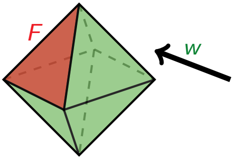
- What can we say about the weight of points in F?

  Every $x, y \in F$ have same weight: $\sum_e w(e)x_e = \sum_e w(e)y_e$

F is the convex hull of $\mathcal{M}$ $\Rightarrow$ every $x, y \in F$ have same weight



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(edge set $\bigcup_{M \in \mathcal{M}} M$)

- Suppose active subgraph has cycle C of $\neq 0$ discepancy

w(green edges) $\neq$ w(red edges)

- Let $x = \frac{1}{|\mathcal{M}|} \sum_{M \in \mathcal{M}} 1_M$ be the mean of the face F
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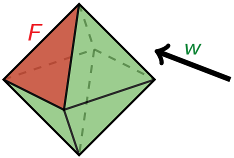

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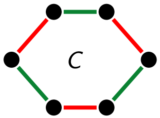

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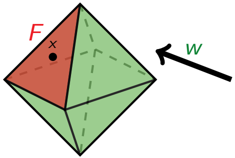

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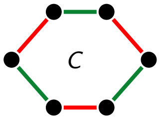

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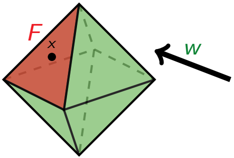

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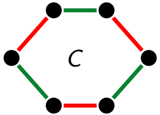

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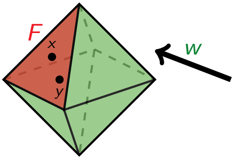

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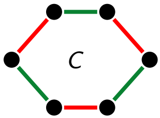

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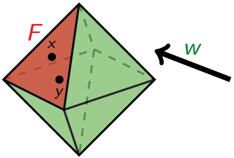

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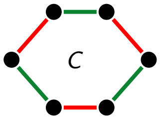

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  - apply $w_1 \in \mathcal{W}$
    - active subgraph has no 4-cycles
- active subgraph has $\leq n^4$ 8-cycles
  - apply $w_2 \in \mathcal{W}$
    - active subgraph has no 8-cycles
- active subgraph has $\leq n^4$ 16-cycles
  - apply $w_3 \in \mathcal{W}$
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- ...
  - apply $w_{\log n} \in \mathcal{W}$
    - active subgraph has no cycles whatsoever
    - success!

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Isolating in stages

$w = w_1$

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Isolating in stages

$$w = \left\langle w_1, w_2, \ldots, w_{\log n} \right\rangle$$

- For each stage i, some $w_i \in \mathcal{W}$ removes the wanted cycles
- So some concatenation $\left\langle w_1, w_2, \ldots, w_{\log n} \right\rangle$ is isolating
- But not sure how to check in $\mathcal{NC}$ if given w_i is good...

The oblivious algorithm checks all concatenations:

$$|\mathcal{W}|^{\log n} = n^{O(\log n)}$$

3. Difficulties of general case & our approach

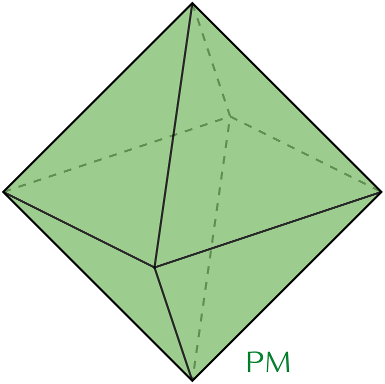
Bipartite key property fails

Bipartite key property

~~Once we assign a cycle $\neq 0$ discrepancy, it will disappear from the active subgraph.~~

Polyhedral perspective

- PM: perfect matching polytope
  (convex hull of all perfect matchings)



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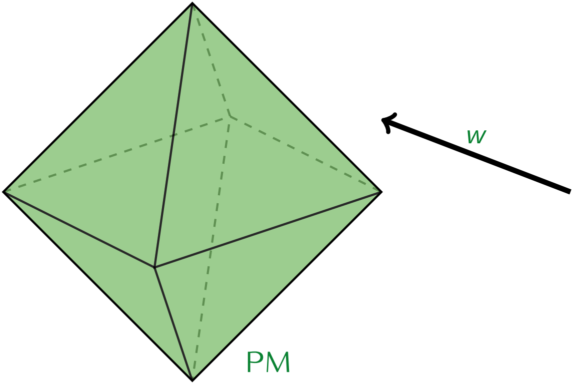

Polyhedral perspective

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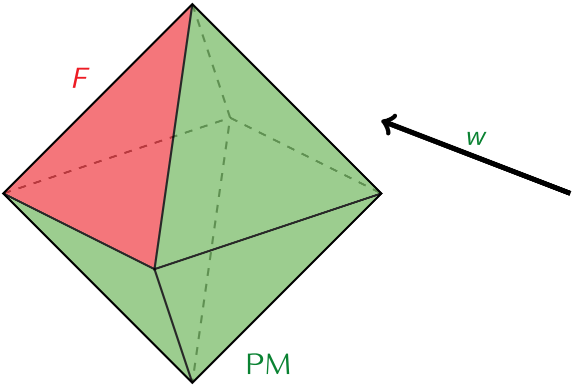
  - F is a face of PM



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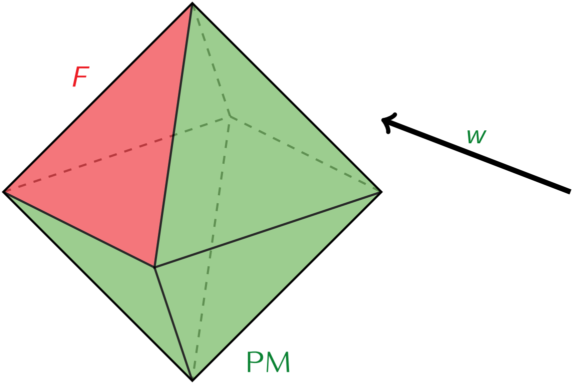

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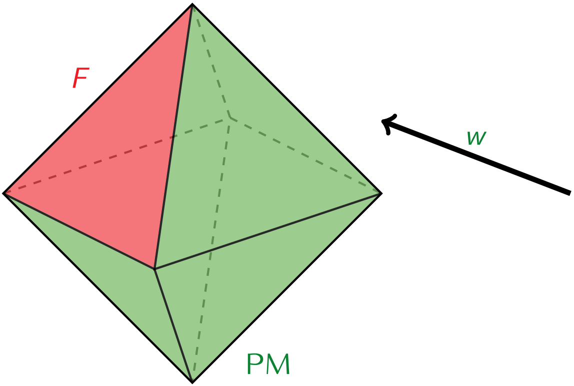

w not isolating

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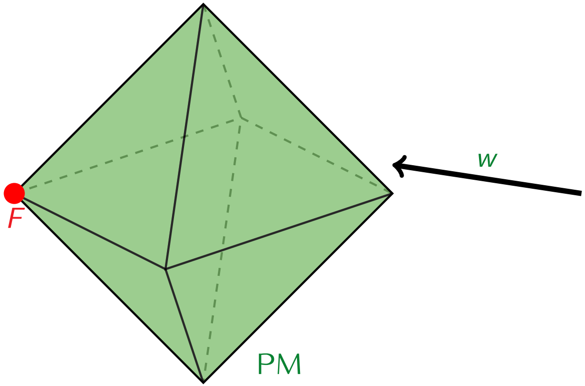

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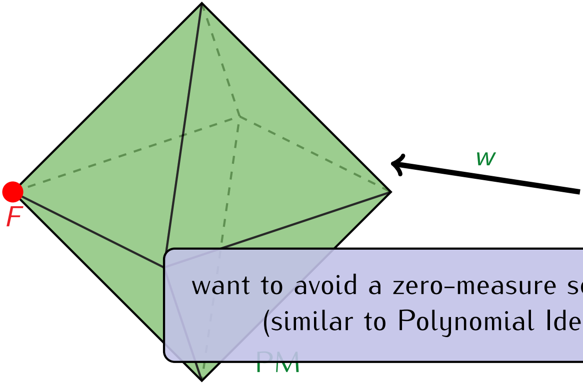
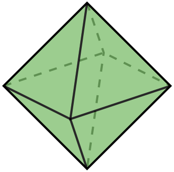


w isolating

want to avoid a zero-measure set deterministically
(similar to Polynomial Identity Testing)

Polyhedral perspective

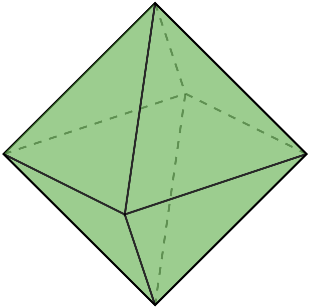
1



isolating in stages

=

decreasing sequence of faces



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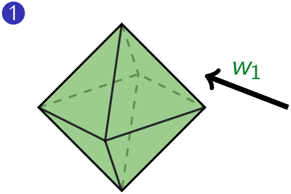

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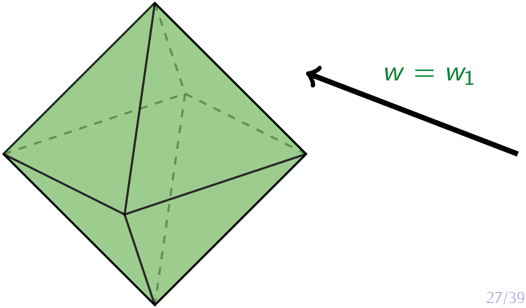

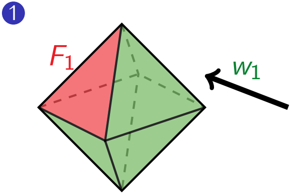

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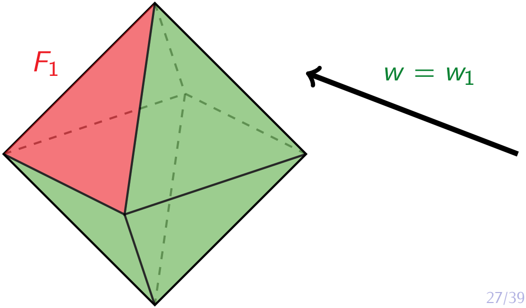

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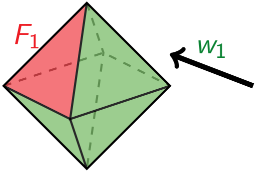

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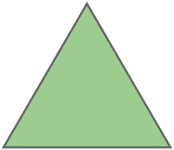

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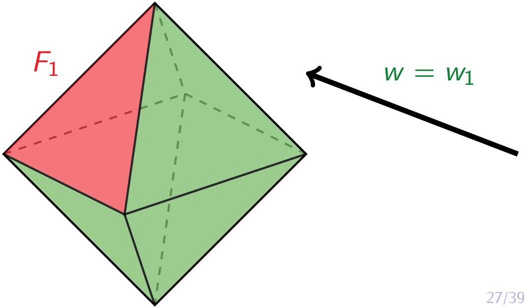

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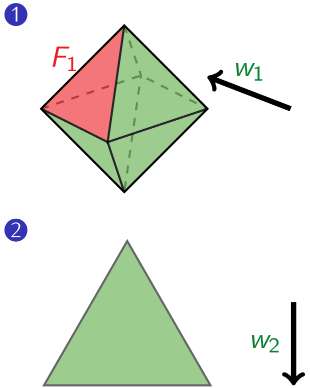

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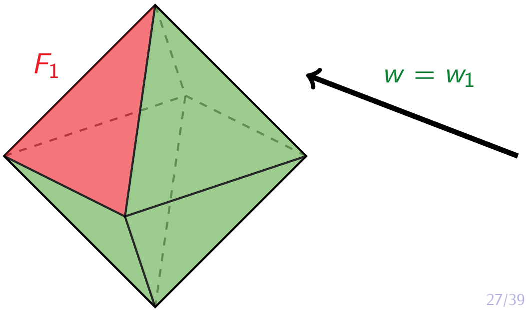

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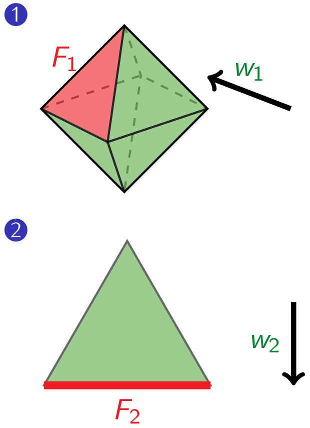

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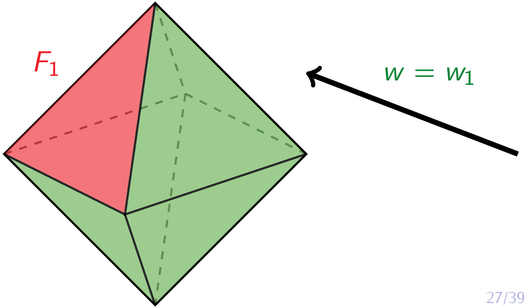

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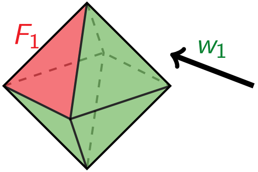

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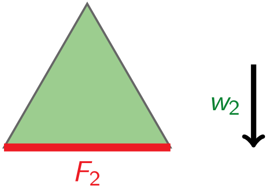

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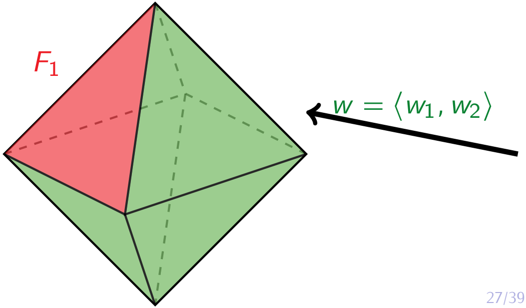

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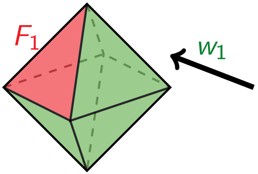

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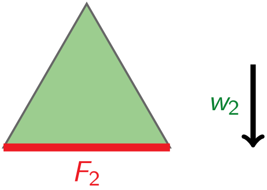

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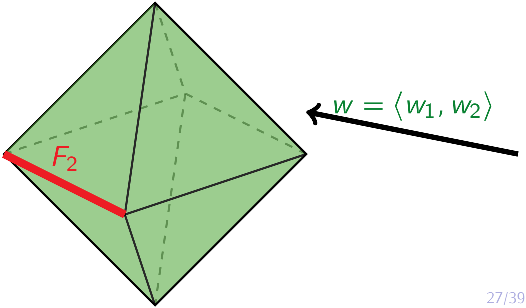

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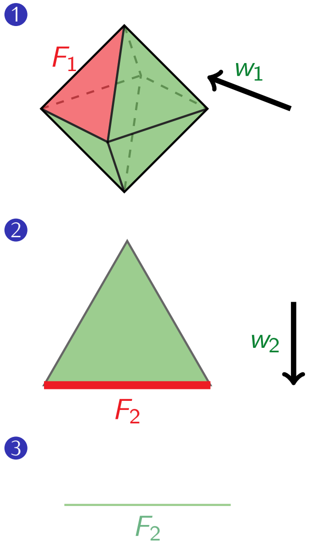

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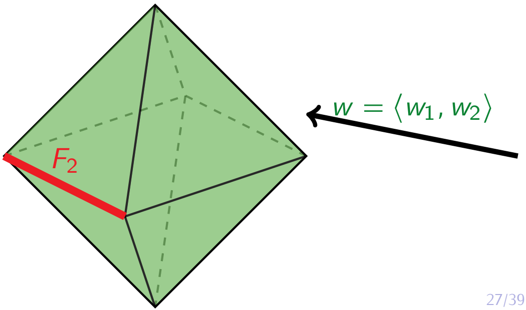

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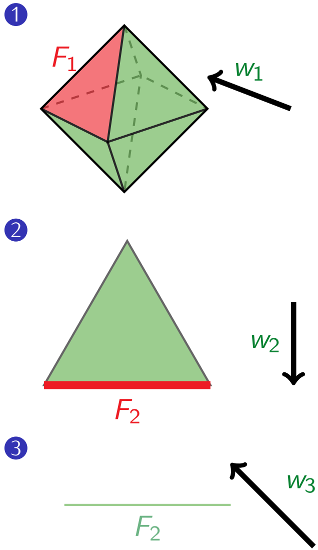

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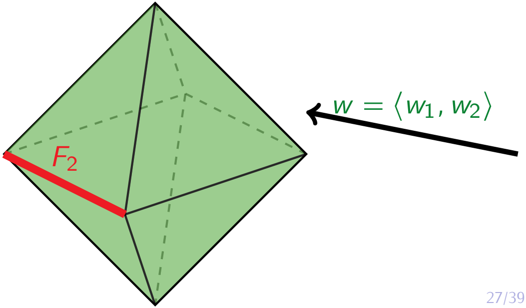

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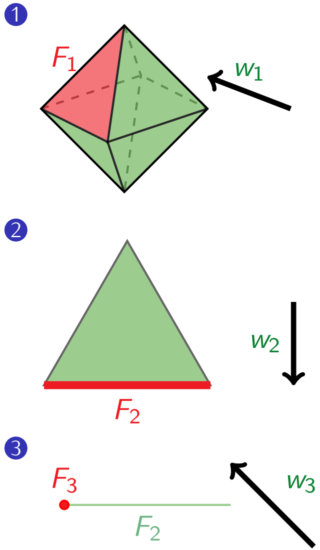

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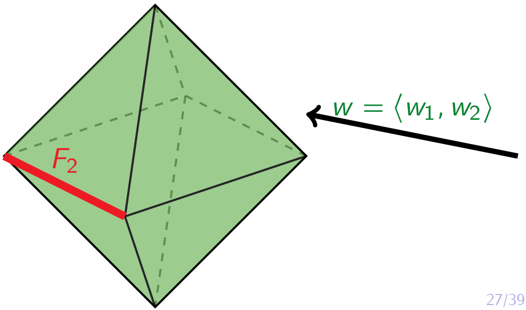

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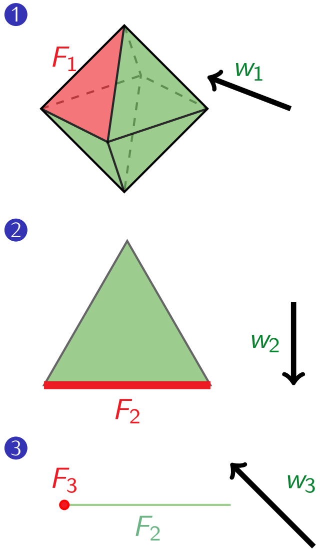

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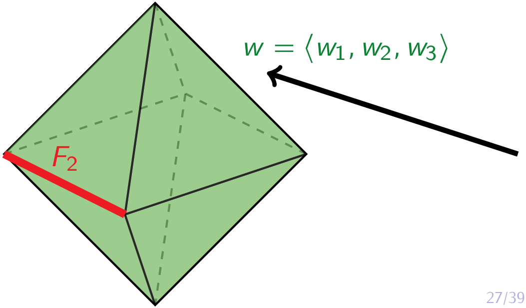
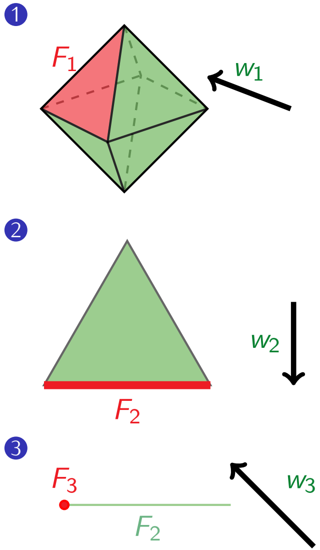
decreasing sequence of faces



Polyhedral perspective



isolating in stages

=

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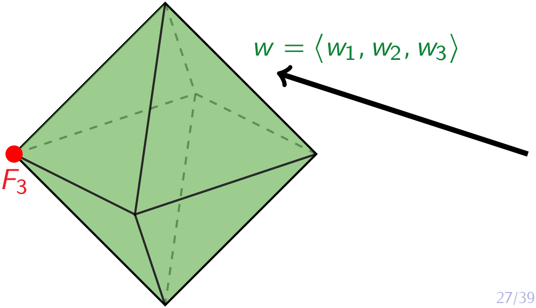

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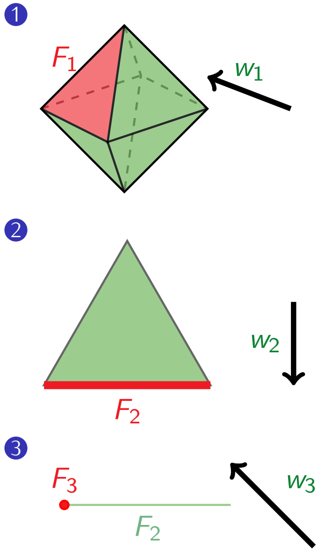

isolating in stages
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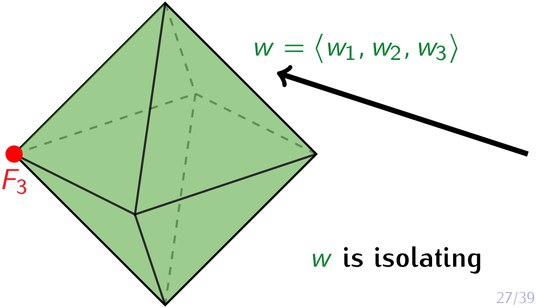
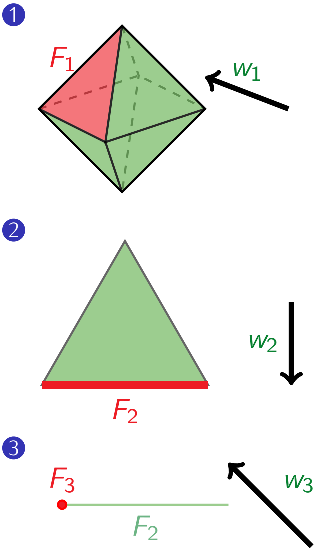


w **is isolating**



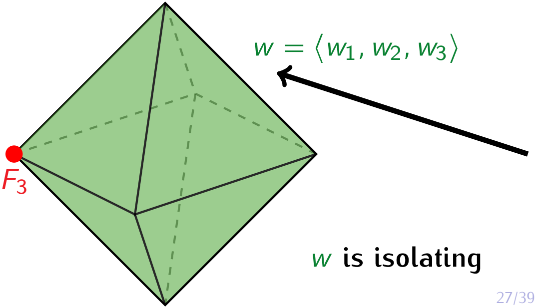
isolating in stages

=

decreasing sequence of faces

decreasing fast due to the bipartite matching polytope:

- bipartite key property: every face is a subgraph
- so girth doubles at every step



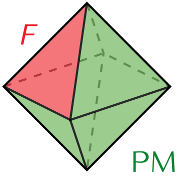
LP formulation

Edmonds [1965]

PM described as set of $x \in \mathbb{R}^E$ such that:

- $x_e \geq 0$ for every edge e
- $x(\delta(v)) = 1$ for every vertex v
- $x(\delta(S)) \geq 1$ for every odd set S of vertices

($\delta(S)$ = edges crossing S)



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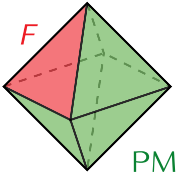

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- In bipartite case:
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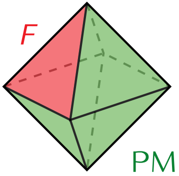
- Now, faces are exponentially harder
- Need $2^{\Omega(n)}$ inequalities [Rothvoss 2013]



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Bipartite key property fails!

$x(\delta(S)) = 1$ for some odd sets $S\}$

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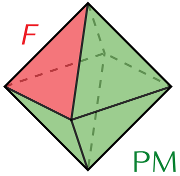
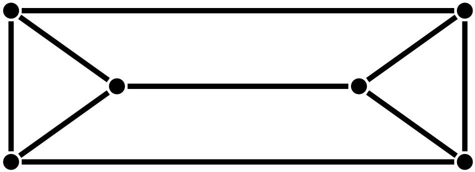


How bipartite key property fails



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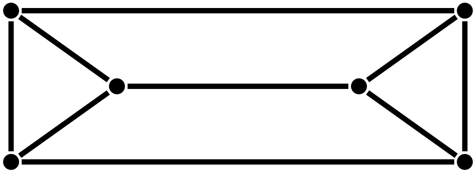
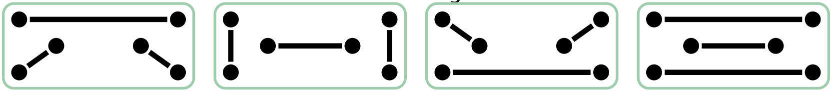


PM: convex hull of all four matchings:



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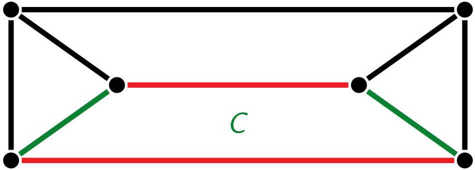
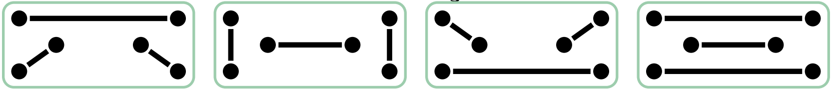


want:
$d_w(C) \neq 0$

PM: convex hull of all four matchings:



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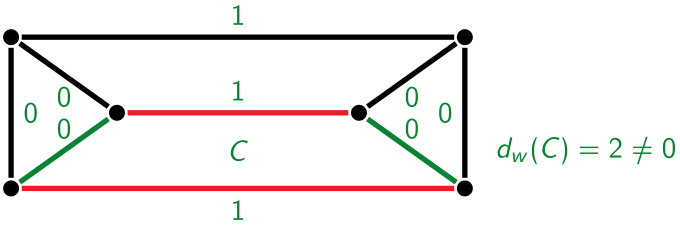

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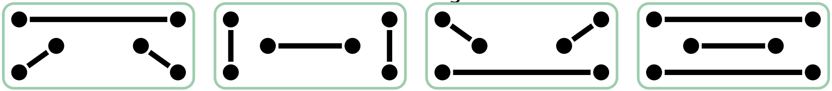

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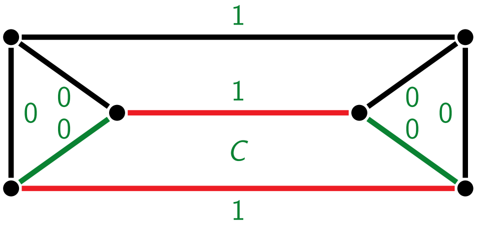

$d_w(C) = 2 \neq 0$

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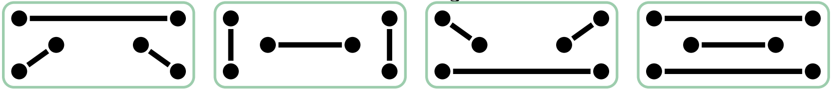

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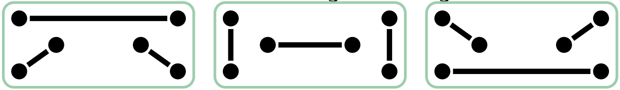

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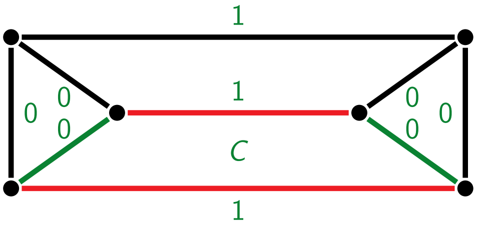

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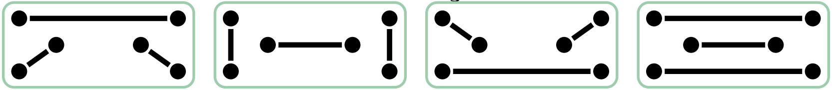

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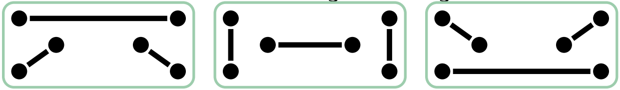

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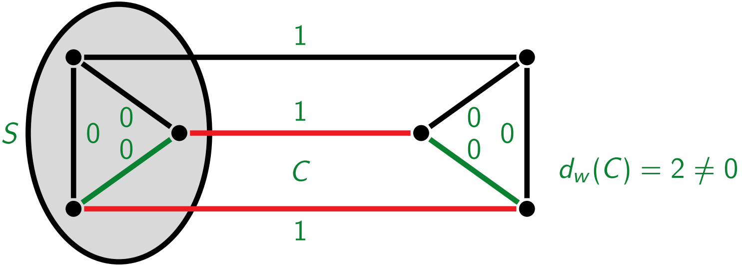

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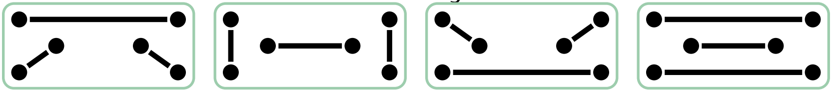

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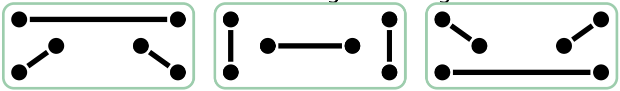

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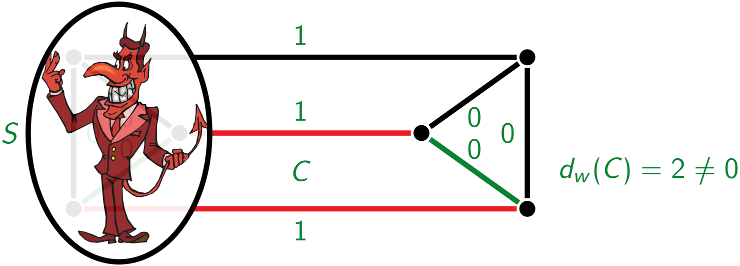

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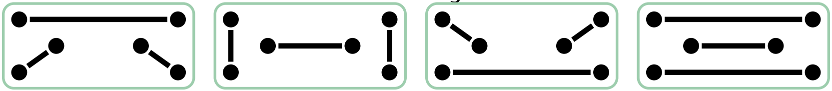

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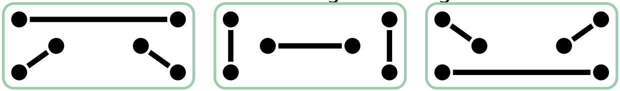

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How we cope

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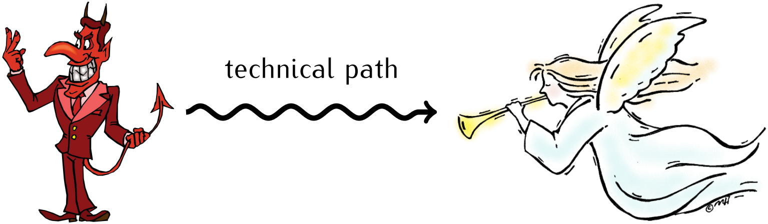

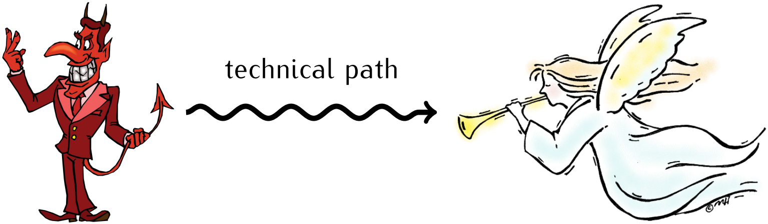

Main ingredients:

- Laminar family of tight cut constraints
- Tight cut constraints decompose the instance
  - $\Rightarrow$ divide-and-conquer approach

Laminarity

Every face F is given as:

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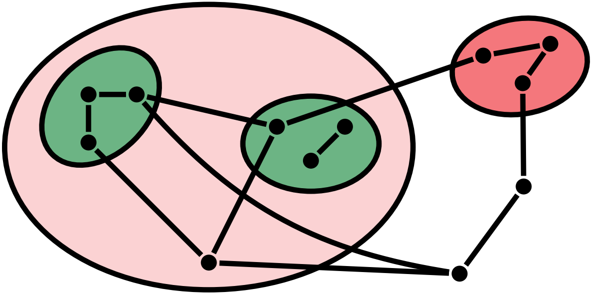
Great news: "some" can be chosen to be a laminar family!

(at most $n/2$ constraints instead of exponentially many to describe a face)



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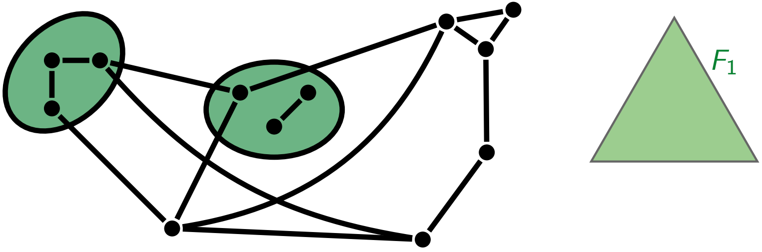

face $\sim$ (edge subset, laminar family)

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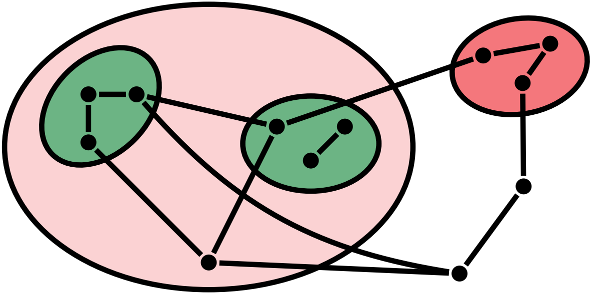

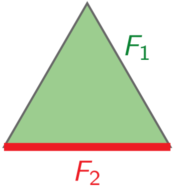

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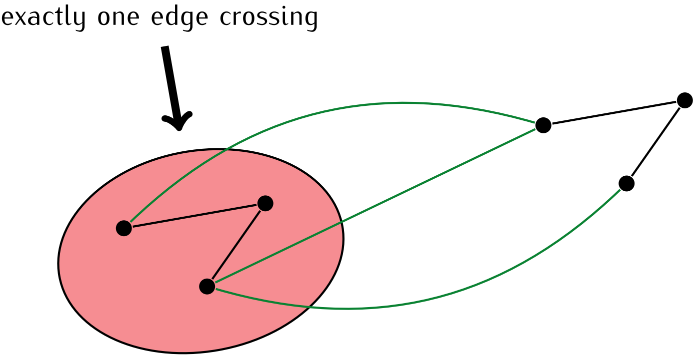
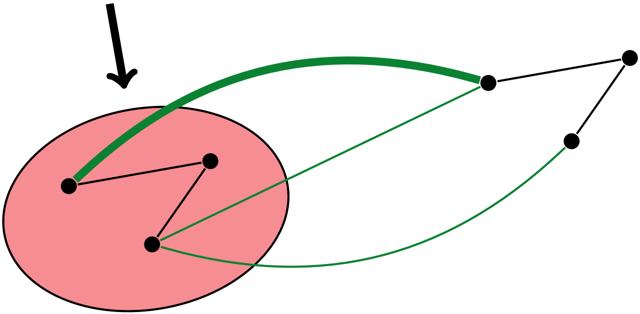
Tight odd cuts are not all bad



- once we fix a boundary edge...

Tight odd cuts are not all bad

exactly one edge crossing



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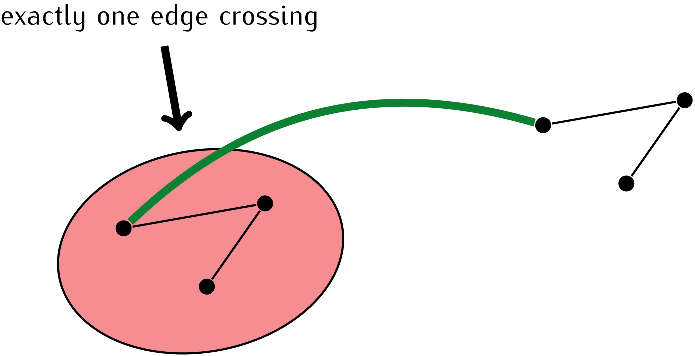

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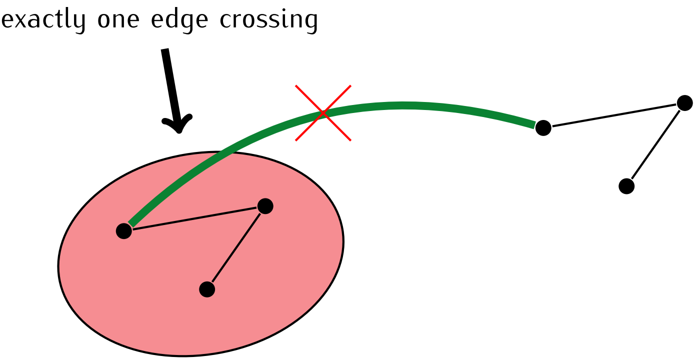

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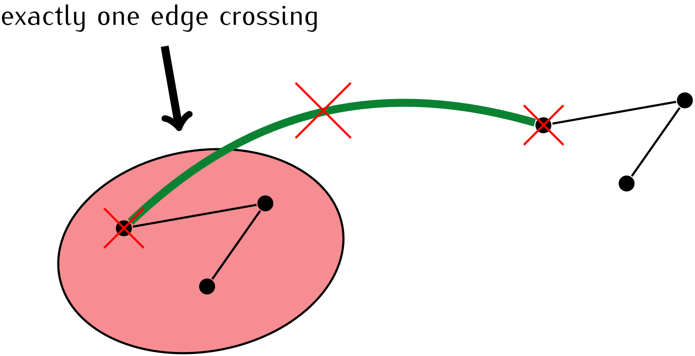

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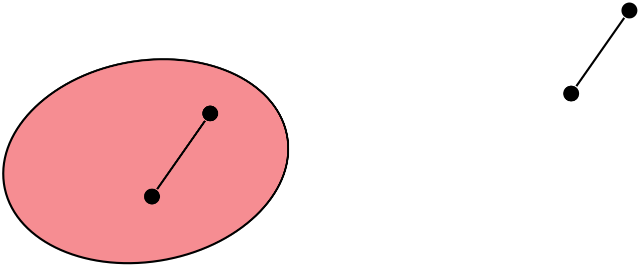

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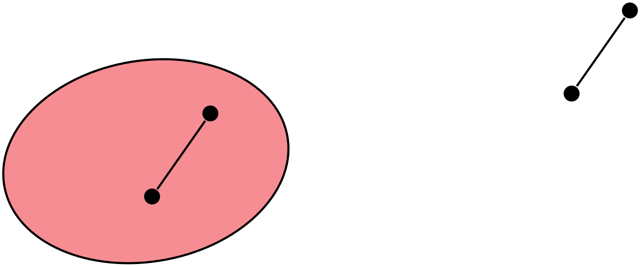

- once we fix a boundary edge...
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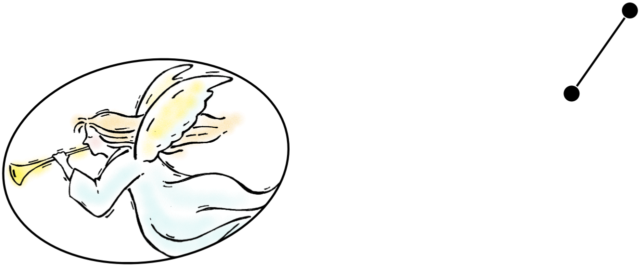

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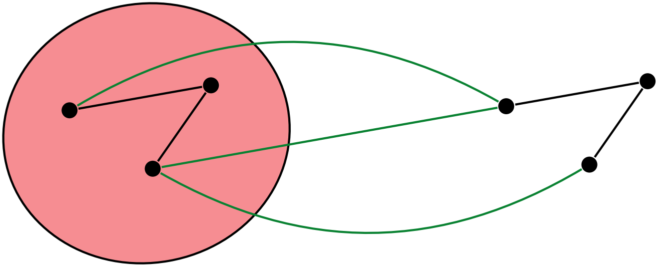
Simplest case of laminar family: only one tight odd set

Between friends: cycles that do not cross tight odd sets behave like in the bipartite case and can thus be removed



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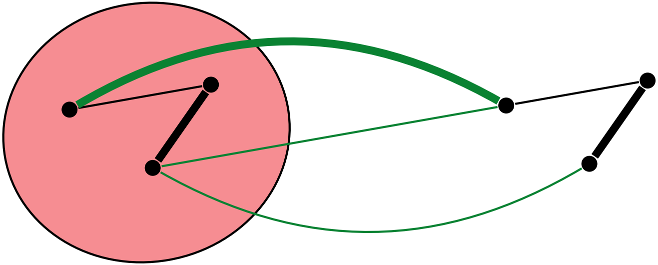

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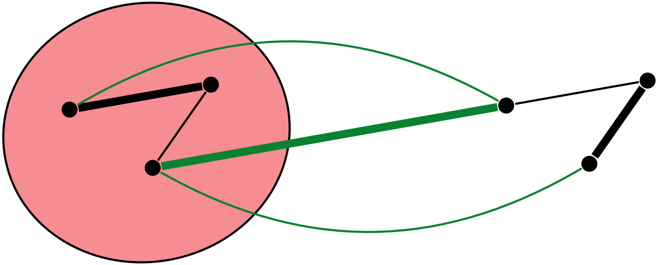

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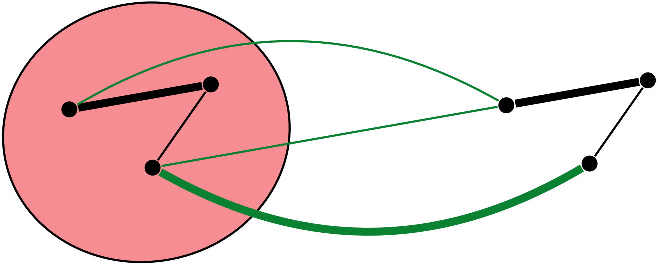

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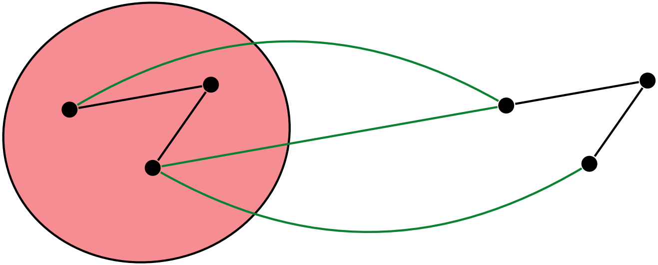

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Divide & conquer

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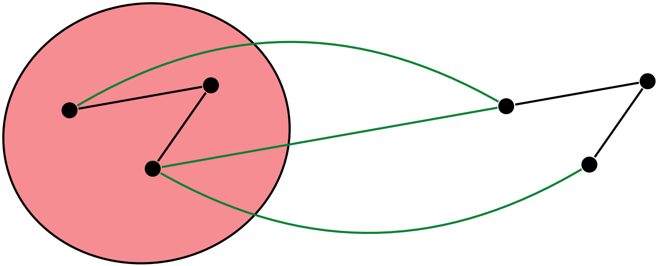
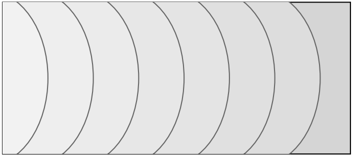


- then every boundary edge determines entire matching
- so: at most n^2 perfect matchings
- some $w \in \mathcal{W}$ will give them different weights

Divide & conquer: chain case



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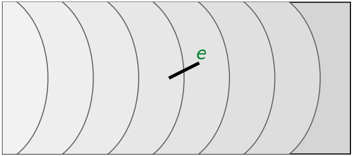

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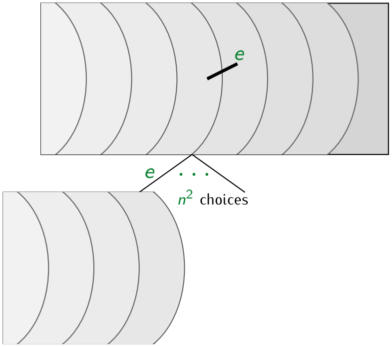

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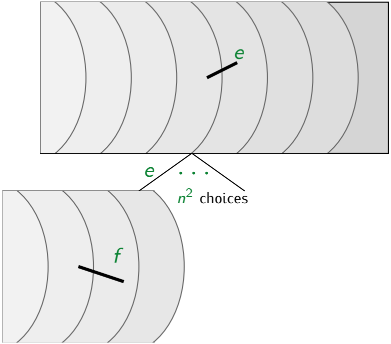

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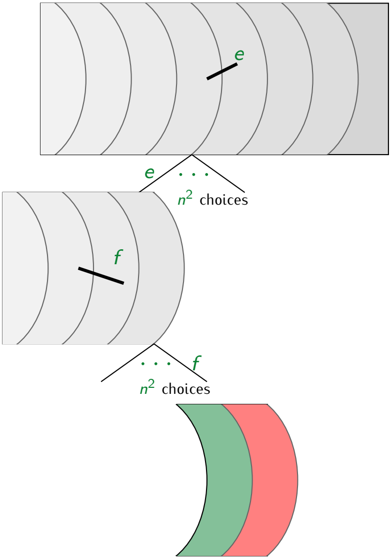

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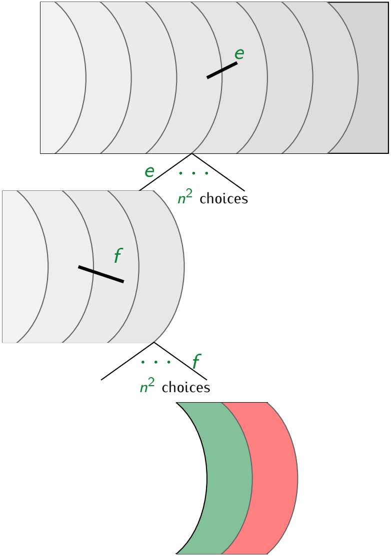

Instance where both sides of the cut are isolated. One $w \in \mathcal{W}'$ makes the entire subinstance isolated

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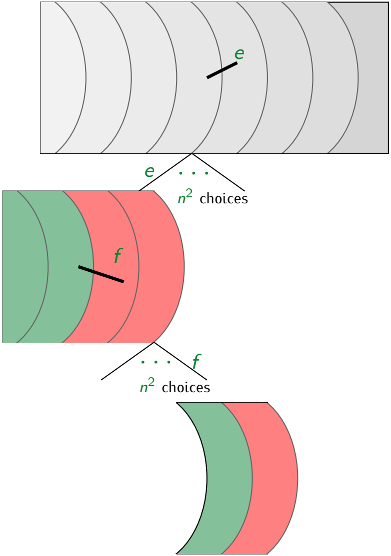

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Divide & conquer: chain case

Now instance where both sides of the cut are isolated. One $w \in \mathcal{W}'$ makes the **entire instance isolated :)**

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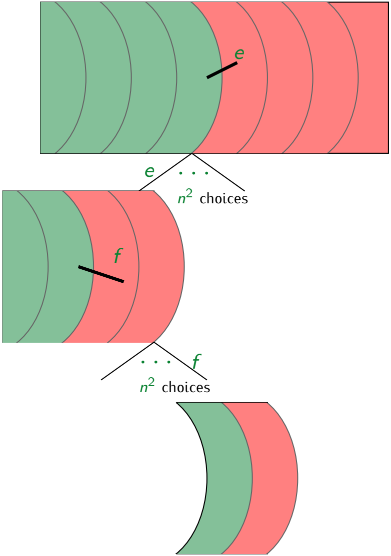

Divide & conquer: chain case

As before we isolate the entire instance in $O(\log n)$ phases

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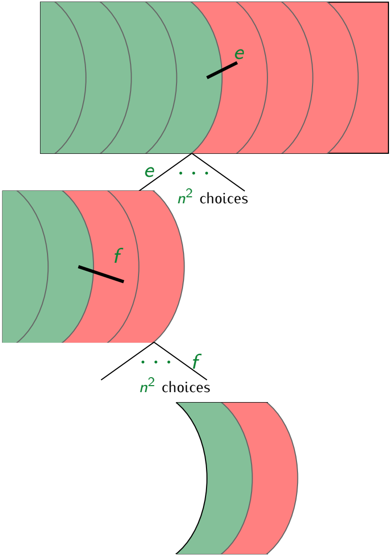

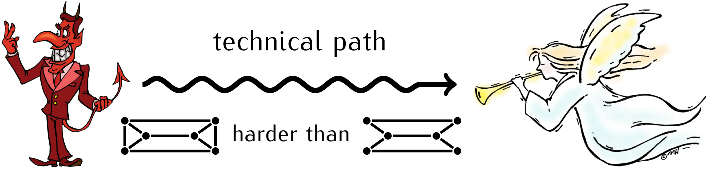

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we reduce the general laminar case to:

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General matching is in QUASI-$\mathcal{NC}$:

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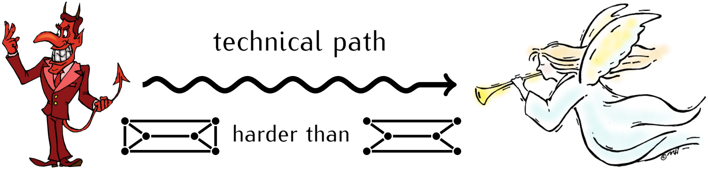

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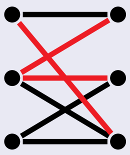

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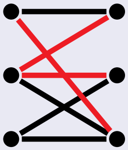

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