Matching is in QUASI-NC

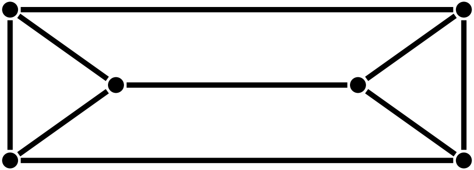
Jakub Tarnawski

joint work with Ola Svensson

September 28, 2017 am Mittag

Perfect matching problem

Given a graph, can we pair up all vertices using edges?



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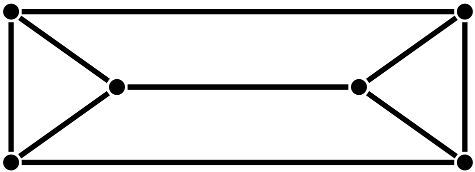
very tough instance:
graph is non-bipartite!



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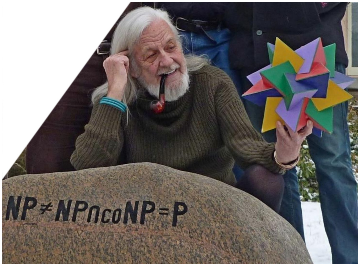
Perfect matching problem

Benchmark problem in computer science

Algorithms:

- bipartite: Jacobi [XIX century, weighted!]
- general: Edmonds [1965]
  - polynomial-time = efficient
- since then, tons of research and still active
- many models of computation: monotone circuits, extended formulations, parallel, distributed, streaming/sublinear, ...



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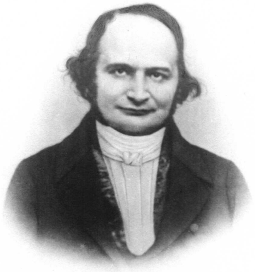

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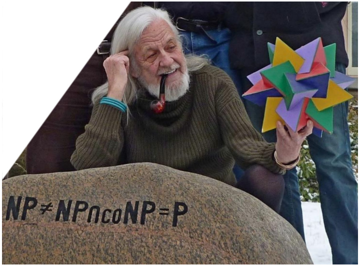

Parallel complexity

Class $\mathcal{NC}$: problems that paralellize completely

poly n processors

poly log n time

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Main open question: is matching in $\mathcal{NC}$?

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it's in **Randomized** $\mathcal{NC}$

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- Matching is in RANDOMIZED $\mathcal{NC}$ [Lovász 1979]:
  has randomized algorithm that uses:
  
  - polynomially many processors
  - polylog time
- Search version is in RANDOMIZED $\mathcal{NC}$:
  
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first matching algorithm
to use Tutte's matrix
and Zippel–Schwartz Lemma

led to understanding of
computational relationship between
search and decision problems

introduced
the Isolation Lemma

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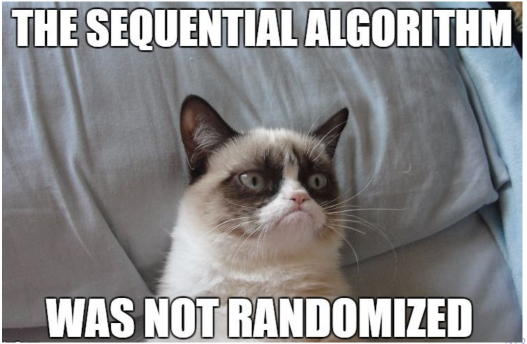

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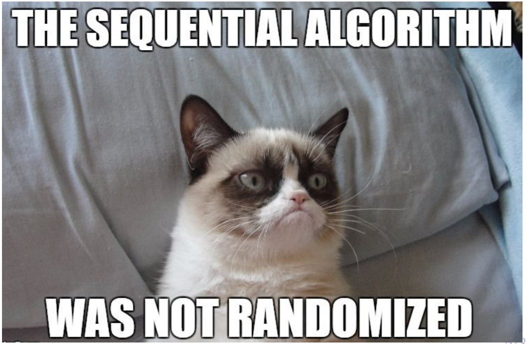

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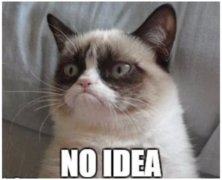

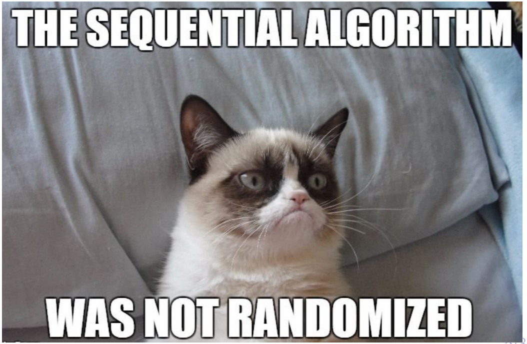

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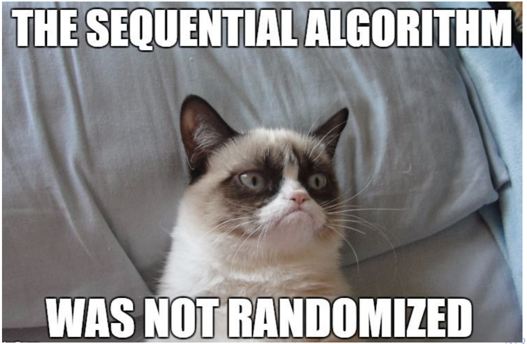

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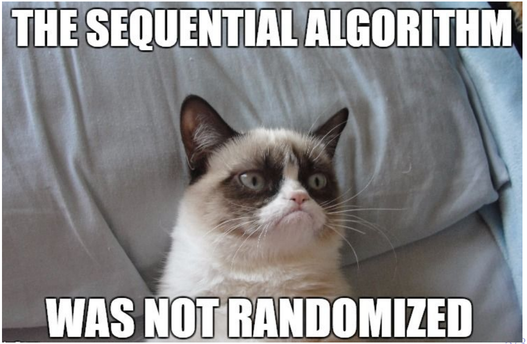

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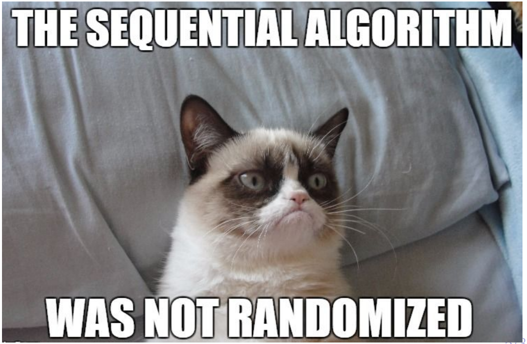

Is matching in $\mathcal{NC}$?

Yes, for restricted graph classes:

- bipartite regular [Lev, Pippenger, Valiant 1981]
- bipartite convex [Dekel, Sahni 1984]
- incomparability graphs [Kozen, Vazirani, Vazirani 1985]
- bipartite graphs with small number of perfect matchings [Grigoriev, Karpinski 1987]
- claw-free [Chrobak, Naor, Novick 1989]
- $K_{3,3}$-free (decision version) [Vazirani 1989]
- planar bipartite [Miller, Naor 1989]
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Is matching in $\mathcal{NC}$?

Fenner, Gurjar and Thierauf [2015] showed:

- Bipartite matching is in QUASI-$\mathcal{NC}$
  ($n^{\text{poly}\log n}$ processors, $\text{poly}\log n$ time, deterministic)



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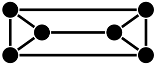
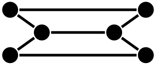
- Approach fails for non-bipartite graphs

 much harder than 

Our result

We show: general matching is in QUASI-$\mathcal{NC}$:

- $n^{\text{poly}\log n}$ processors
- $\text{poly}\log n$ time
- deterministic

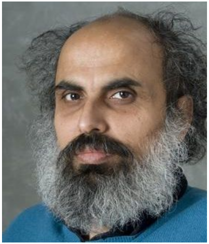
Outline

1. Isolating weight functions
   [Mulmuley, Vazirani, Vazirani 1987]

2. Bipartite case
   [Fenner, Gurjar, Thierauf 2015]

3. Difficulties of general case
   & our approach

1. Isolating weight functions [Mulmuley, Vazirani, Vazirani 1987]



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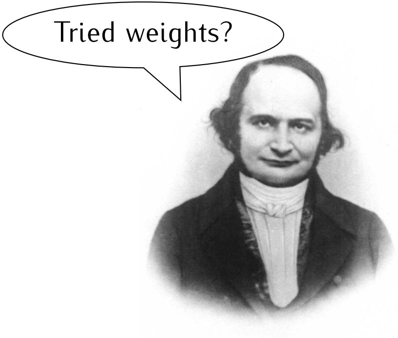

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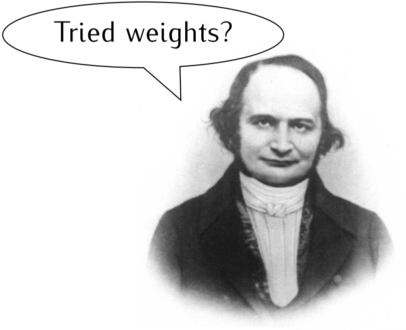

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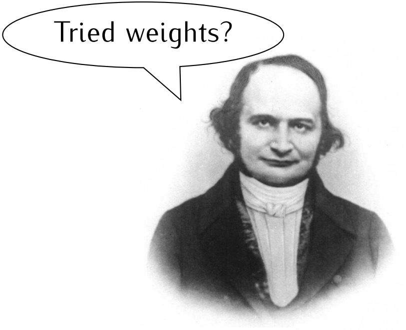

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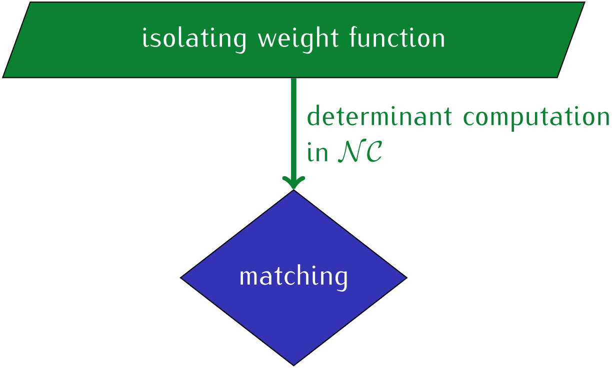
Weight function $w : E \to \mathbb{Z}_+$ is **isolating**
if there is a **unique** min-weight perfect matching

[Mulmuley, Vazirani, Vazirani 1987]



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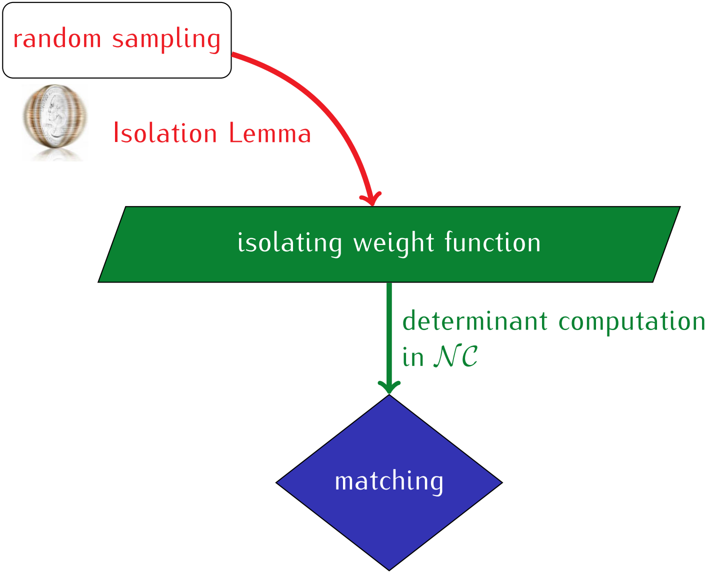

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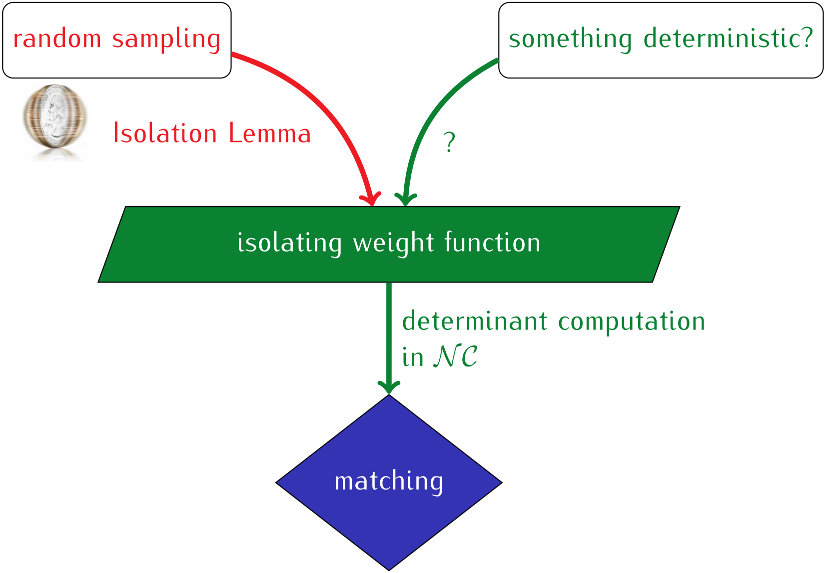

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Isolation Lemma [MVV 1987]

If each $w(e)$ picked randomly from $\{1, 2, \ldots, n^3\}$,
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- holds more generally,
  for any set family in place of matchings!
- many applications in complexity theory
- related to Polynomial Identity Testing

Derandomize the Isolation Lemma

- **Challenge**:
  get an isolating weight function
  deterministically in $\mathcal{NC}$

- **We prove**:
  can construct $n^{O(\log^2 n)}$ weight functions in QUASI-$\mathcal{NC}$
  such that one of them is isolating

- We do it without looking at the graph

- **Implies: matching is in QUASI-$\mathcal{NC}$**

Special case of derandomizing Polynomial Identity Testing
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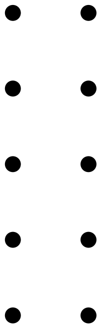
2. Bipartite case [Fenner, Gurjar, Thierauf 2015]

Goal: how to construct $n^{O(\log n)}$ weight functions such that one of them is isolating?

Isolating matchings

What if w is **not** isolating?

- there are perfect matchings M, M' with $w(M) = w(M')$ minimum



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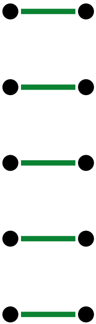

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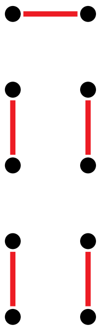

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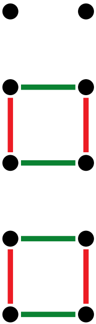

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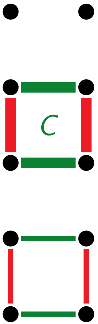

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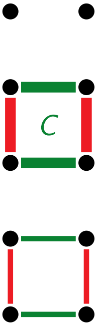

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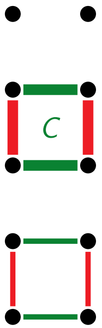

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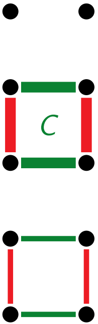

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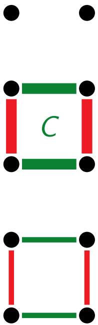

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Removing cycles

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Lemma

There is a poly-sized set $\mathcal{W}$ of weight functions such that:
for any n^4 cycles,
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Not so easy, but we can cope with all **4-cycles**.

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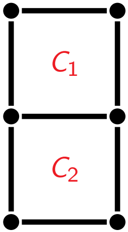

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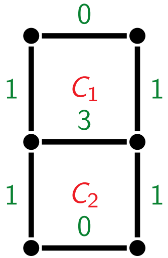

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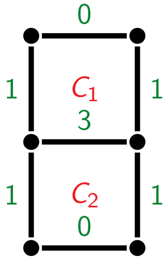
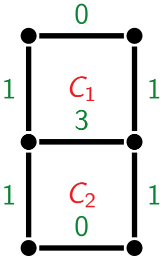


$d_w(C_1) = 1 \neq 0$
$d_w(C_2) = 1 \neq 0$

Active subgraph:
those edges that are in a min-weight perfect matching



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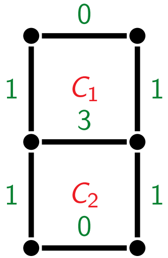

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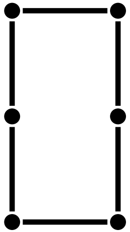

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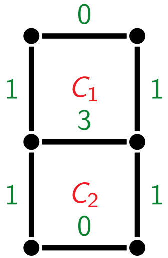
Bipartite key property

Once we assign a cycle $\neq 0$ discrepancy,
it will disappear from the active subgraph.



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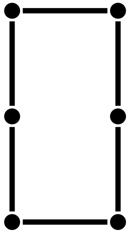

Removing cycles

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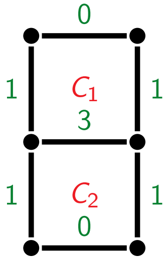
That is, any perfect matching in the active subgraph is min-weight.



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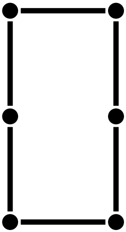

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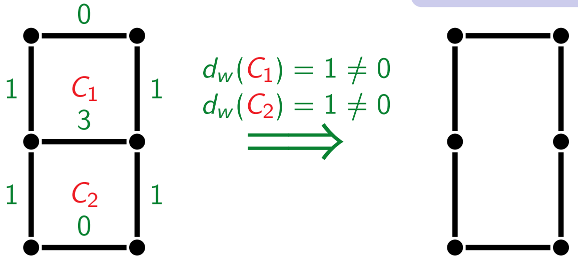

By assigning $\neq 0$ discrepancy to 4-cycles, we can remove them.
Then continue restricted to the smaller active subgraph!

Isolating in stages

Lemma

There is a poly-sized set $\mathcal{W}$ of weight functions such that:
for any n^4 cycles,
some $w \in \mathcal{W}$
assigns **all of them** $\neq 0$ discrepancy.

- active subgraph has $\leq n^4$ 4-cycles
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Isolating in stages

$$w = \left\langle w_1, w_2, \ldots, w_{\log n} \right\rangle$$

- For each stage i, some $w_i \in \mathcal{W}$ removes the wanted cycles
- So some concatenation $\left\langle w_1, w_2, \ldots, w_{\log n} \right\rangle$ is isolating
- But not sure how to check in $\mathcal{NC}$ if given w_i is good...

The oblivious algorithm checks all concatenations:

$$|\mathcal{W}|^{\log n} = n^{O(\log n)}$$

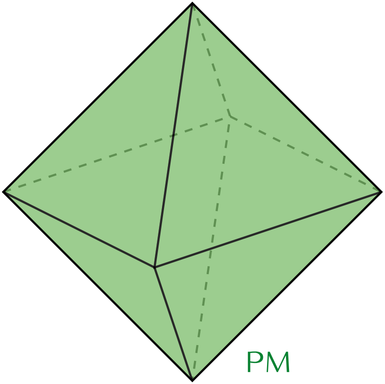
3. Difficulties of general case & our approach

Bipartite key property fails

~~**Bipartite key property**~~

~~Once we assign a cycle $\neq 0$ discrepancy, it will disappear from the active subgraph.~~

- PM: perfect matching polytope
  (convex hull of all perfect matchings)



Polyhedral perspective

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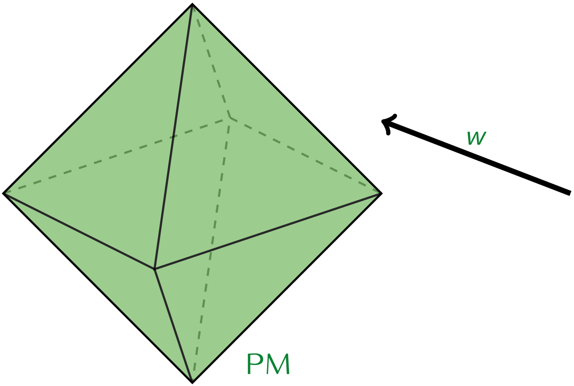

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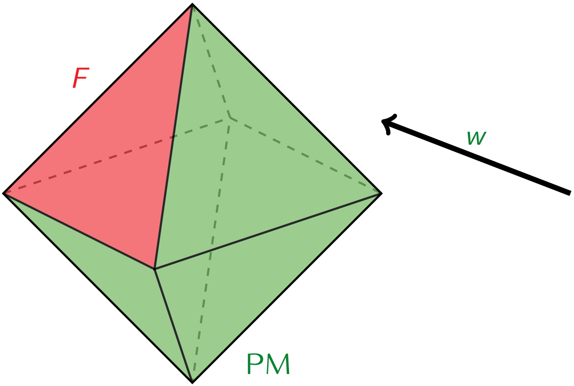
  - F is a face of PM



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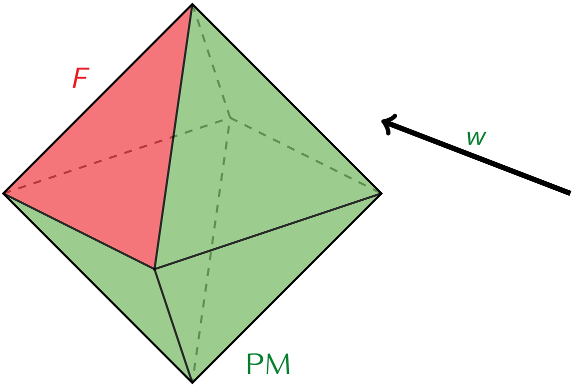

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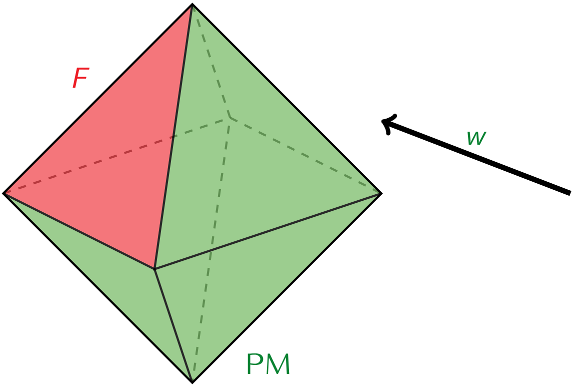

w not isolating

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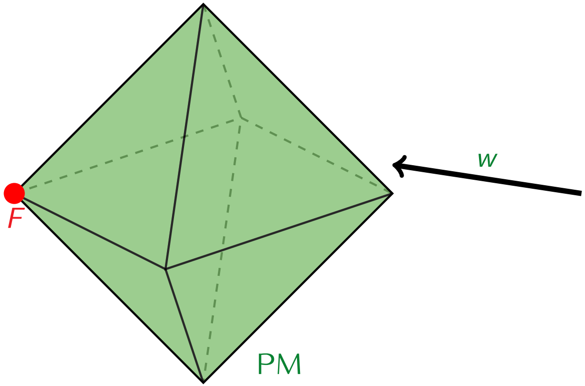

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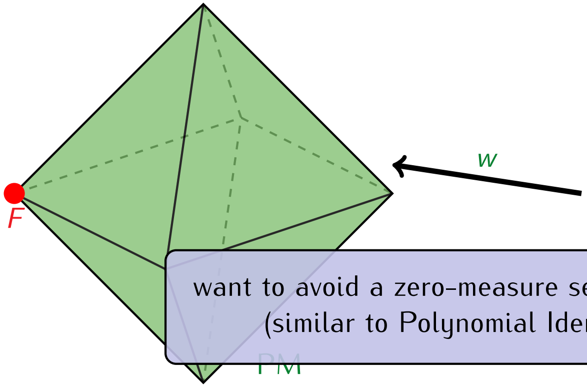
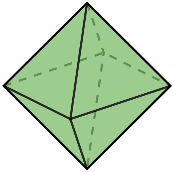


w isolating

want to avoid a zero-measure set deterministically
(similar to Polynomial Identity Testing)

Polyhedral perspective

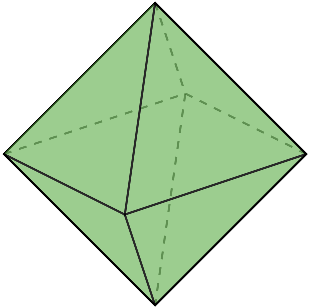
1



isolating in stages

=

decreasing sequence of faces



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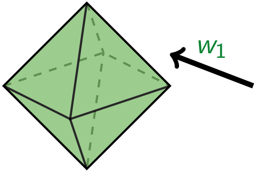

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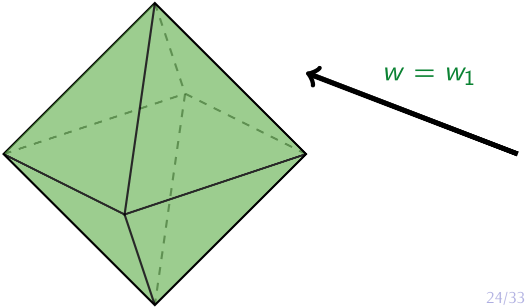

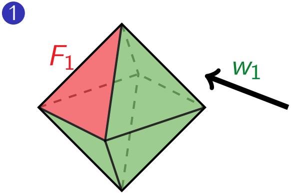

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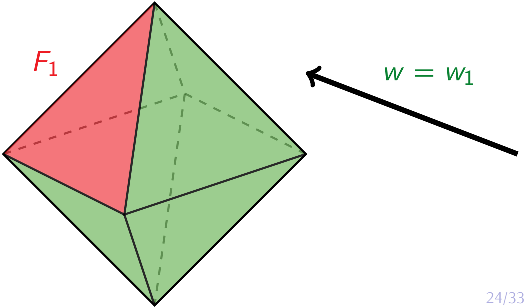

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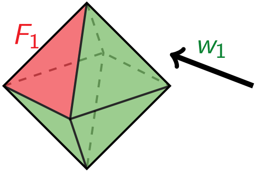

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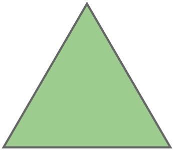

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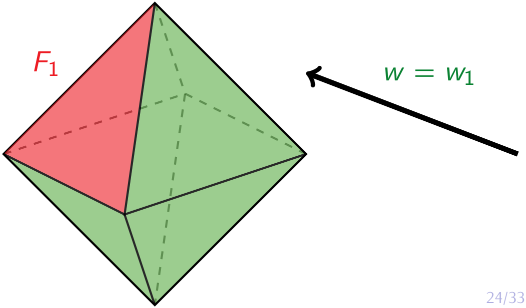

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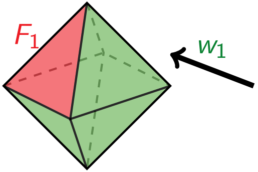

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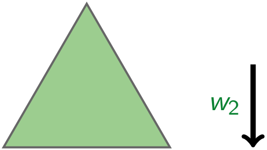

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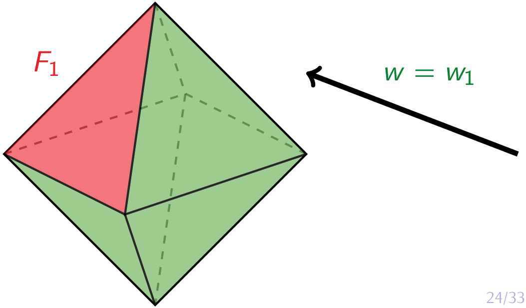

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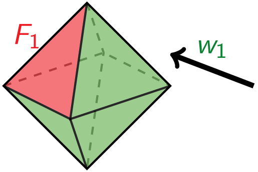

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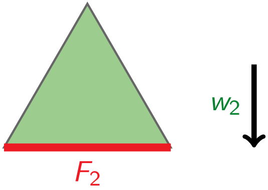
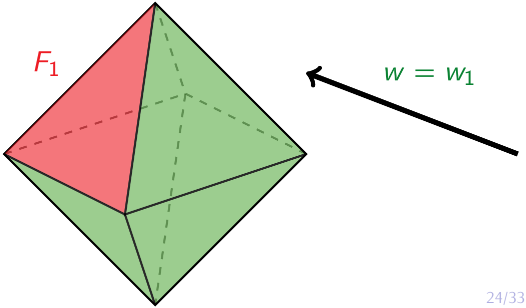


isolating in stages

=

decreasing sequence of faces



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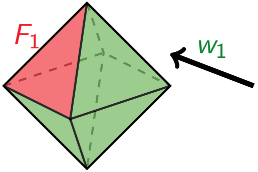

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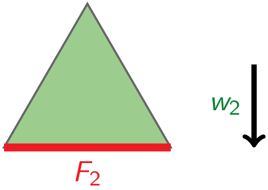

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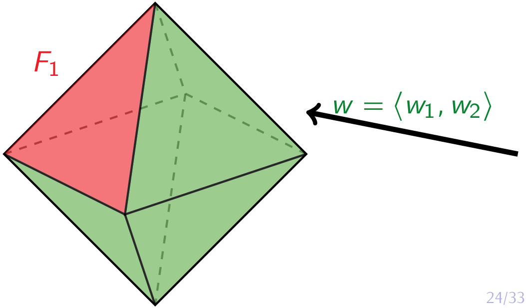
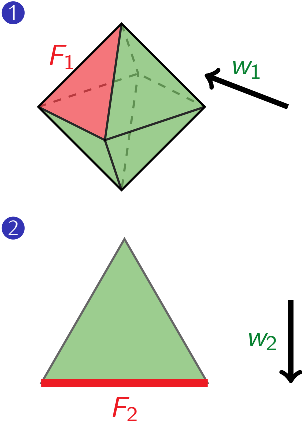


Polyhedral perspective



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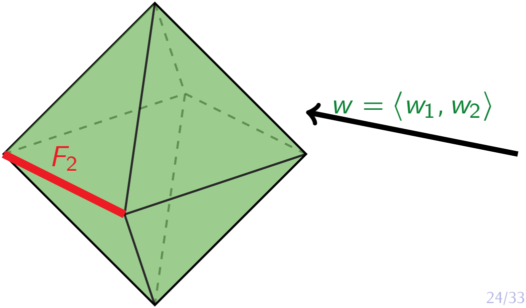

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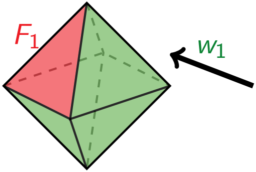

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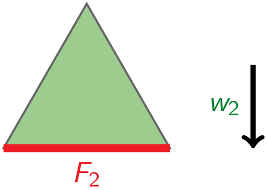

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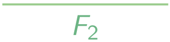

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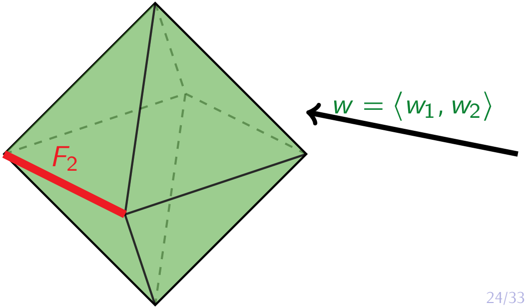

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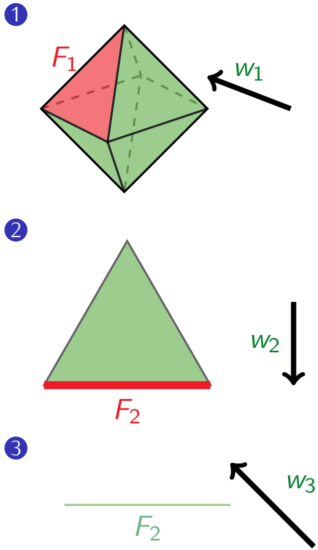

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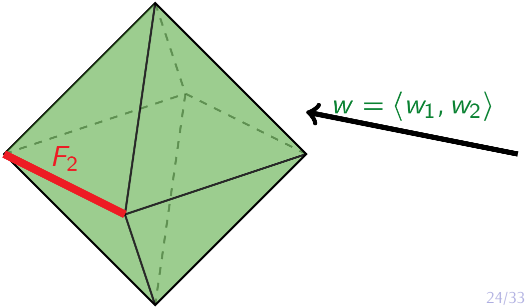

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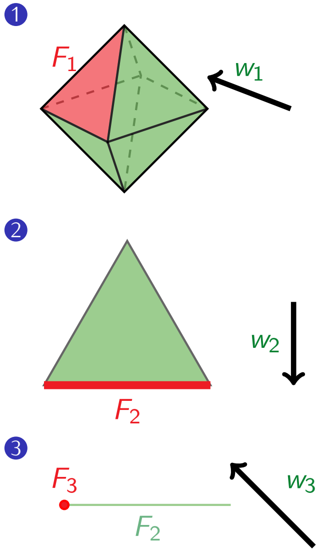

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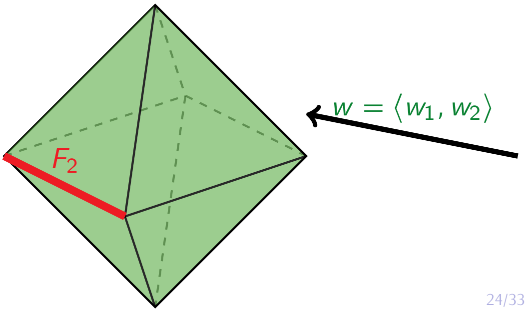

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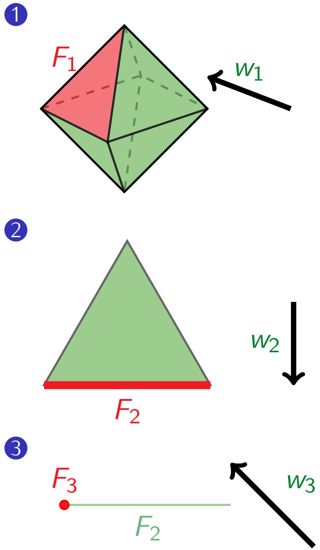

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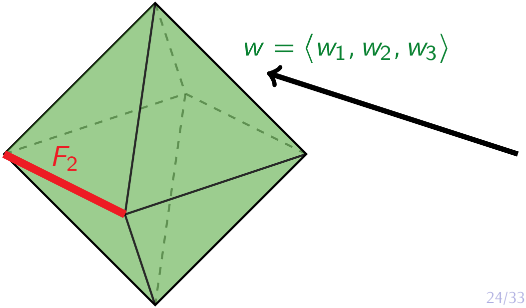

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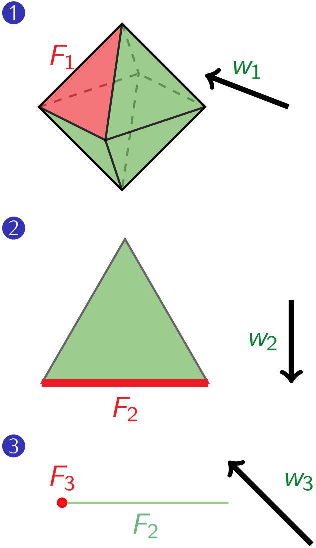

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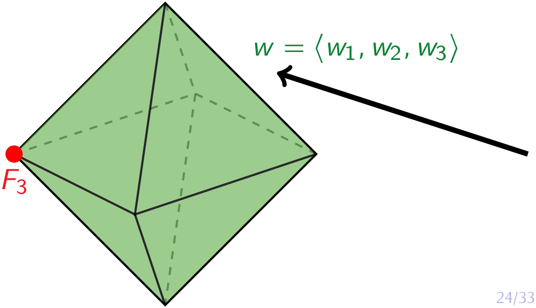

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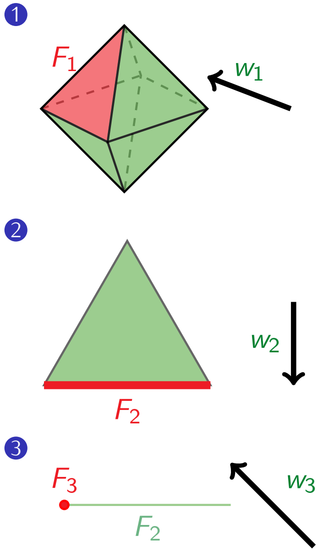

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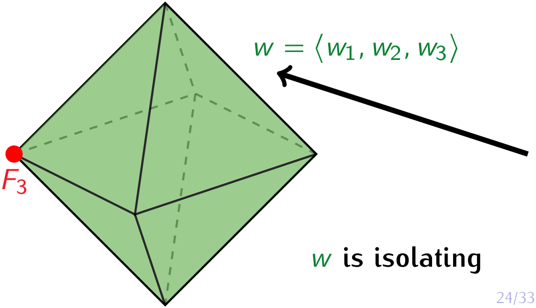

w **is isolating**

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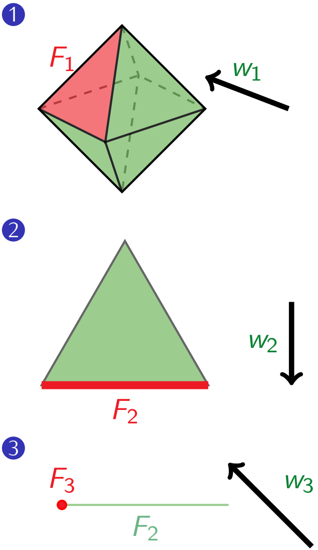


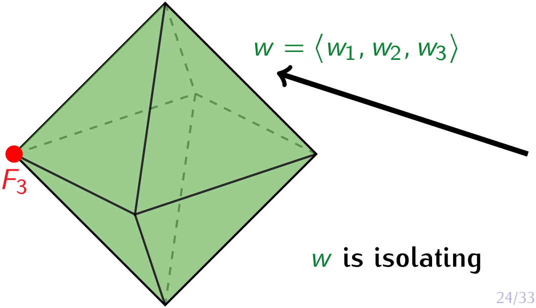
isolating in stages

=

decreasing sequence of faces

decreasing fast due to the bipartite matching polytope:

- bipartite key property: every face is a subgraph
- so girth doubles at every step



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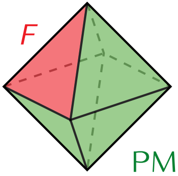
LP formulation

Edmonds [1965]

PM described as set of $x \in \mathbb{R}^E$ such that:

- $x_e \geq 0$ for every edge e
- $x(\delta(v)) = 1$ for every vertex v
- $x(\delta(S)) \geq 1$ for every odd set S of vertices

($\delta(S)$ = edges crossing S)



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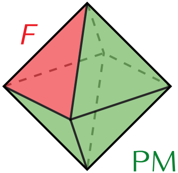

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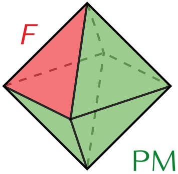
- In bipartite case:
  $F = \{x \in \text{PM} : \; x_e = 0 \;$ for some edges $e\}$
  (F given by the active subgraph)
- Now, faces are exponentially harder
- Need $2^{\Omega(n)}$ inequalities [Rothvoss 2013]



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Bipartite key property fails!

$x(\delta(S)) = 1$ for some odd sets $S\}$

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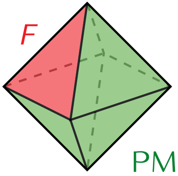
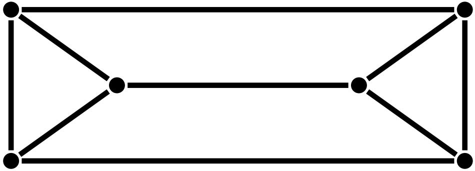


How bipartite key property fails



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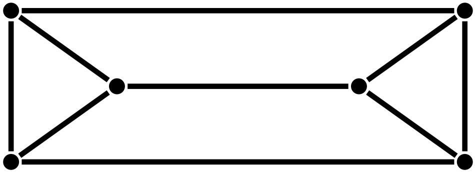
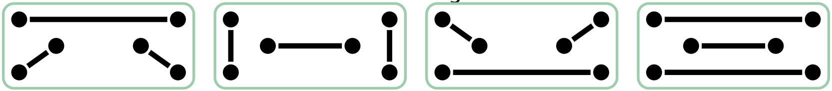


PM: convex hull of all four matchings:



How bipartite key property fails

want:
$d_w(C) \neq 0$

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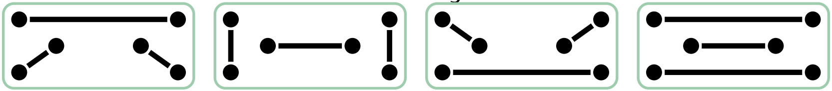

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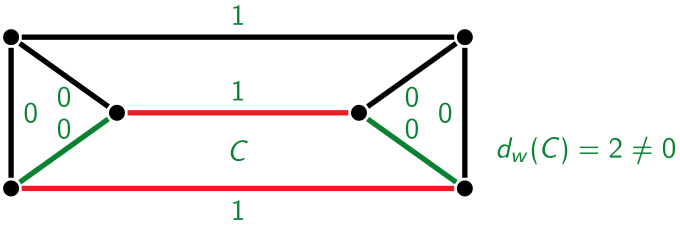

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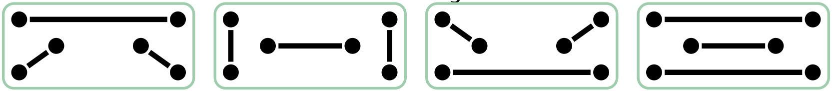

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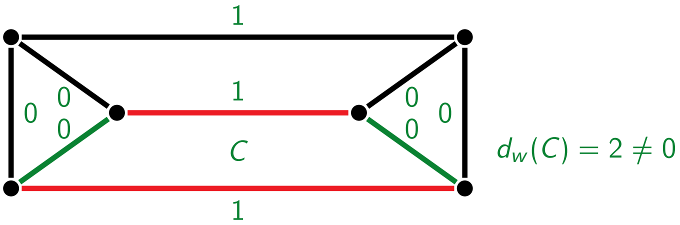

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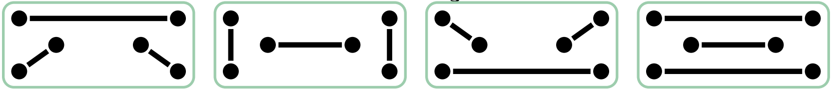
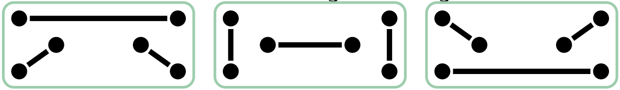


F: convex hull of matchings of weight 1:



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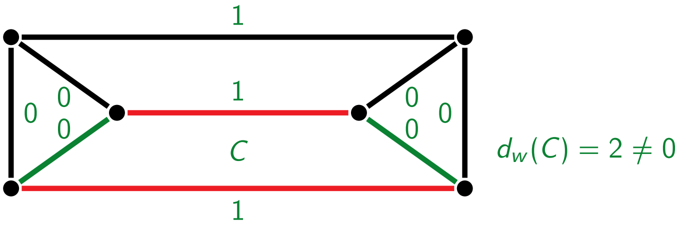

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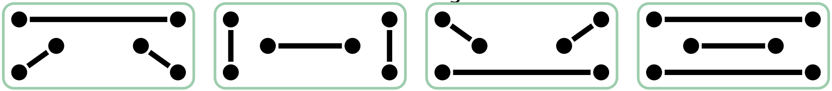

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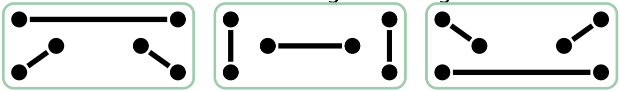
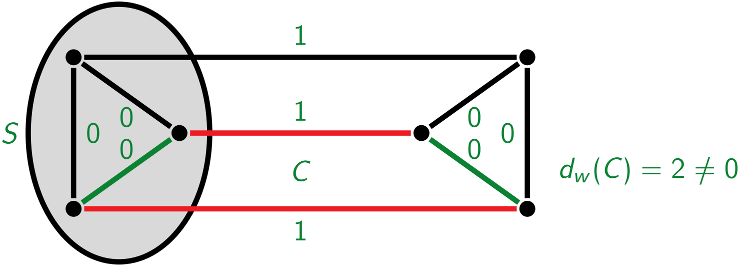


$F \subsetneq PM$ but still has all edges... 😢

How bipartite key property fails



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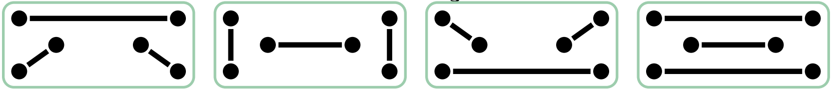

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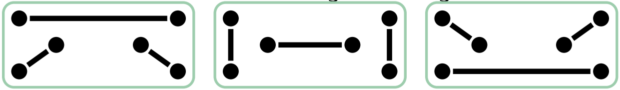

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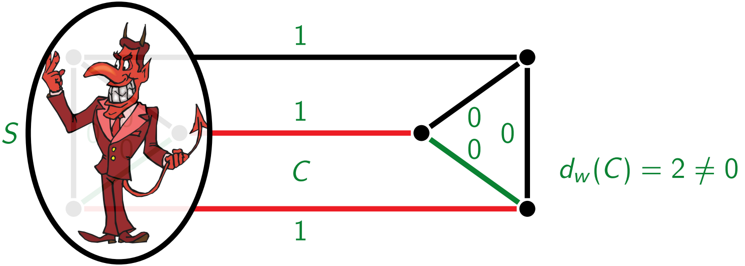

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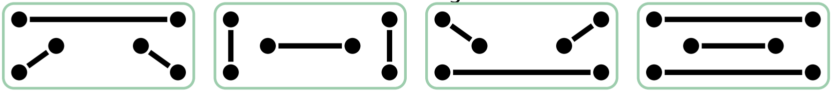

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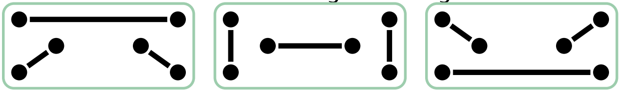

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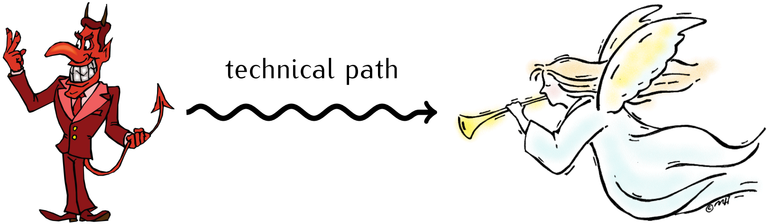
How we cope

How we cope

technical path

How we cope



Main ingredients:

- Laminar family of tight cut constraints
- Tight cut constraints decompose the instance
  - $\Rightarrow$ divide-and-conquer approach

Every face F is given as:

$$F = \{x \in \mathrm{PM} : \begin{array}{ll} x_e = 0 & \text{for some edges } e, \\ x(\delta(S)) = 1 & \text{for some odd sets } S\} \end{array}$$

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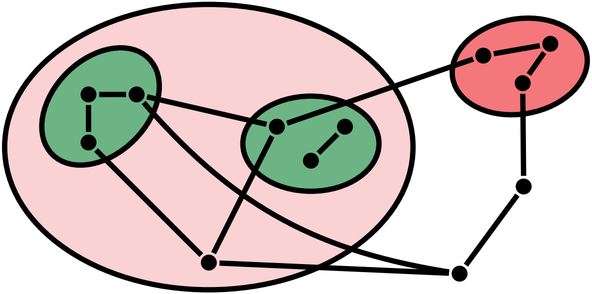
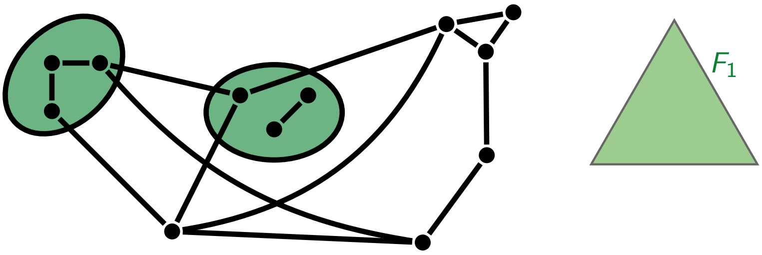
Great news: “some” can be chosen to be a laminar family!

(at most $n/2$ constraints instead of exponentially many to describe a face)



Laminarity



face $\sim$ (edge subset, laminar family)

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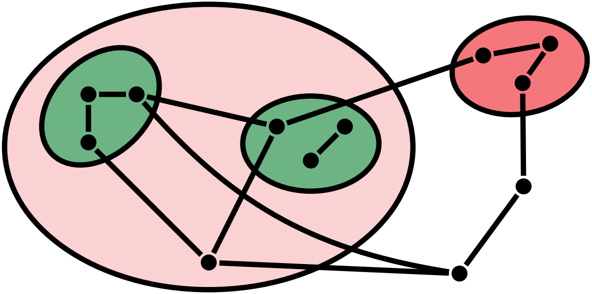

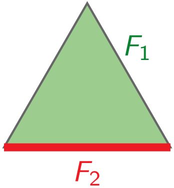

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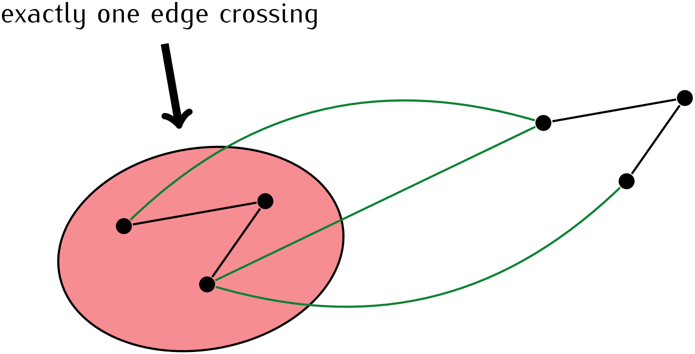
Tight odd cuts are not all bad



- once we fix a boundary edge...

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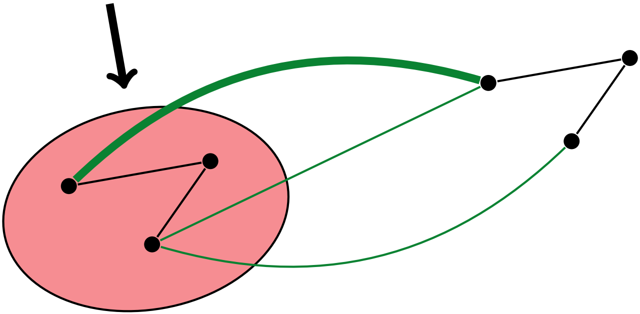

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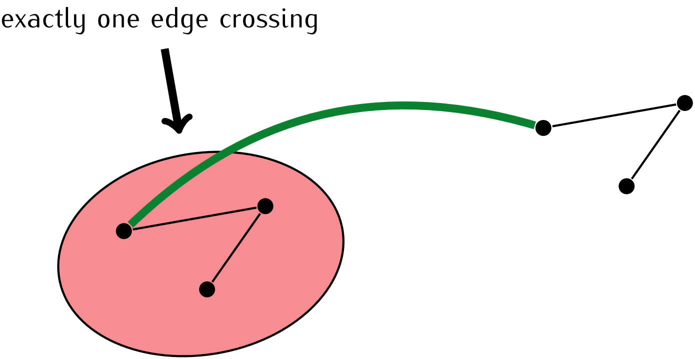

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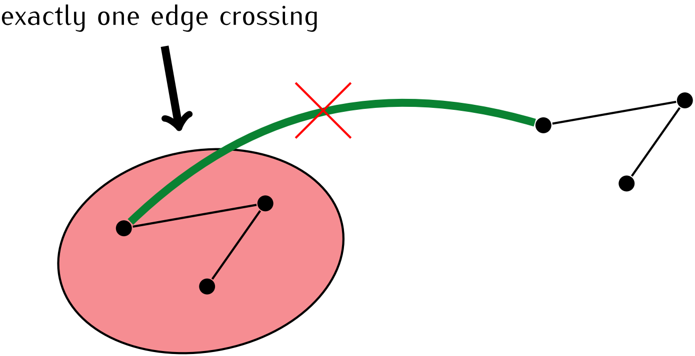

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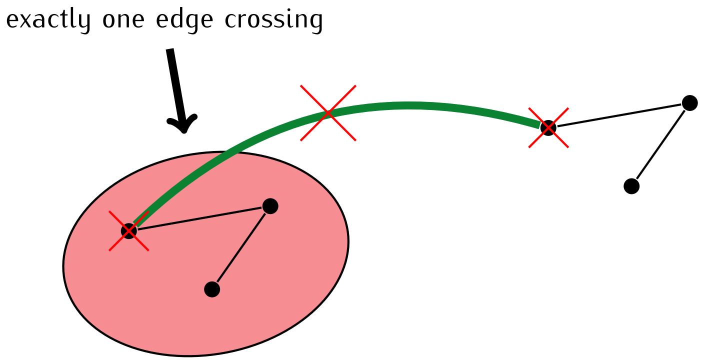

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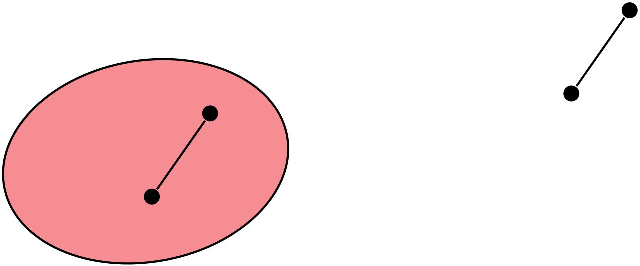

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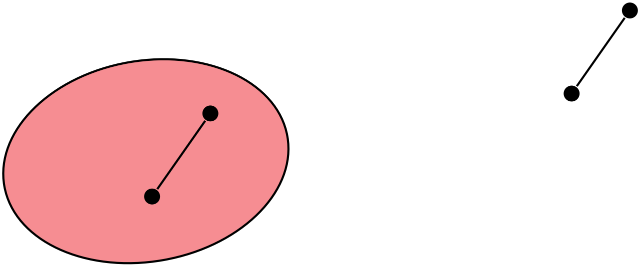

- once we fix a boundary edge...
- ... the instance decomposes into two **independent** ones

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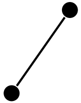

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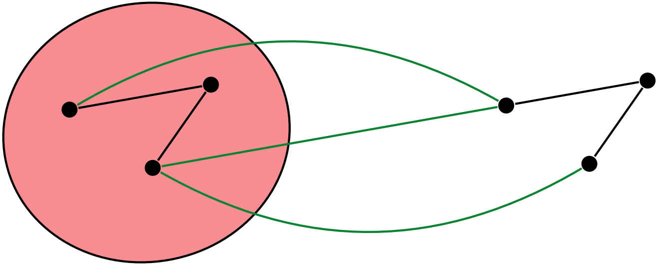
Divide & conquer

Simplest case of laminar family: only one tight odd set

Between friends: cycles that do not cross tight odd sets behave like in the bipartite case and can thus be removed



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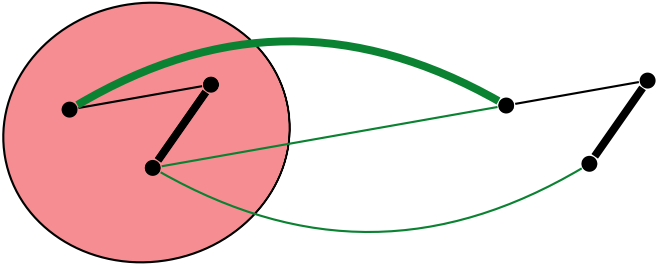

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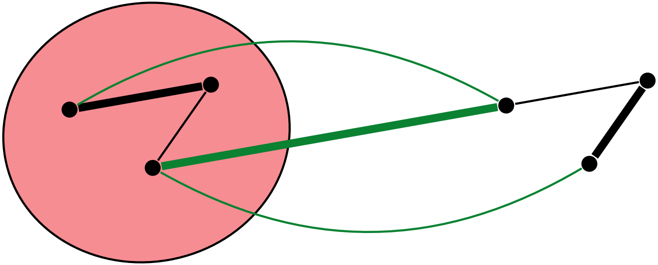

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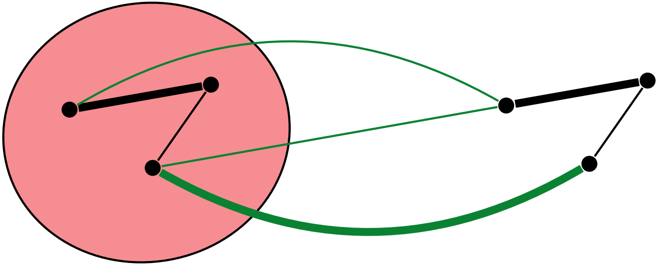

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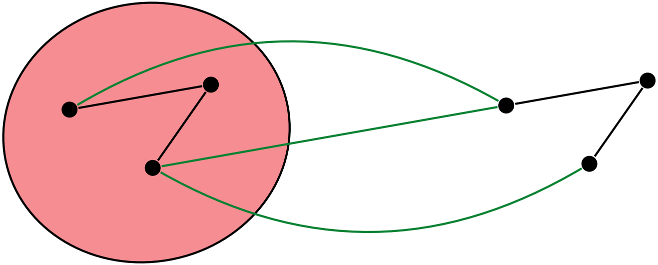

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Divide & conquer

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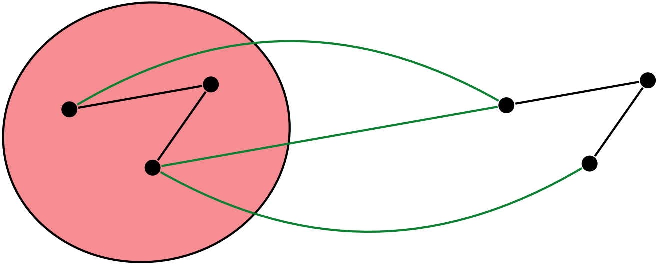

- then every boundary edge determines entire matching
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- some $w \in \mathcal{W}$ will give them different weights

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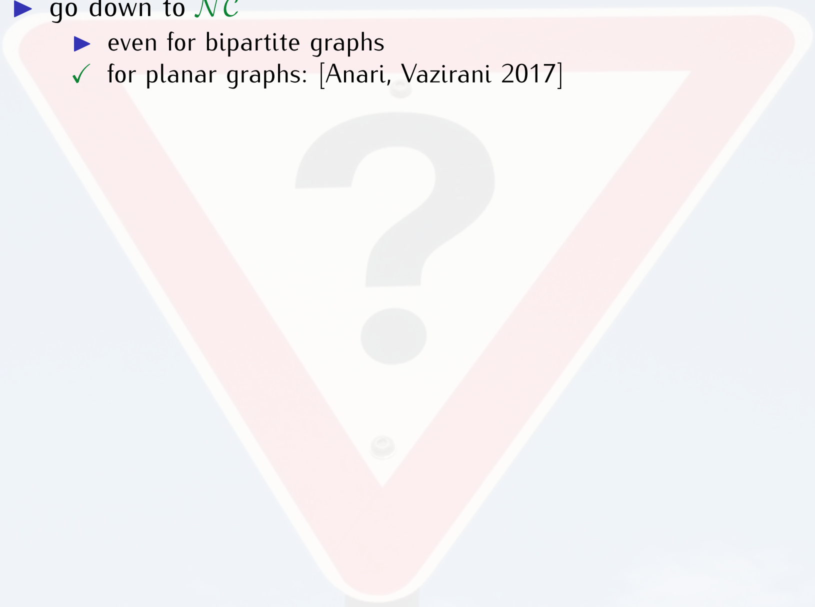
Details: see paper or talk to me :)

Future work

- go down to $\mathcal{NC}$
  - even for bipartite graphs
  - ✓ for planar graphs: [Anari, Vazirani 2017]



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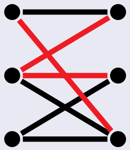

Given: graph with some edges red, number k.
Is there a perfect matching with exactly k red edges?

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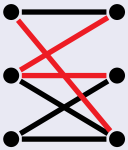

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Thank you!