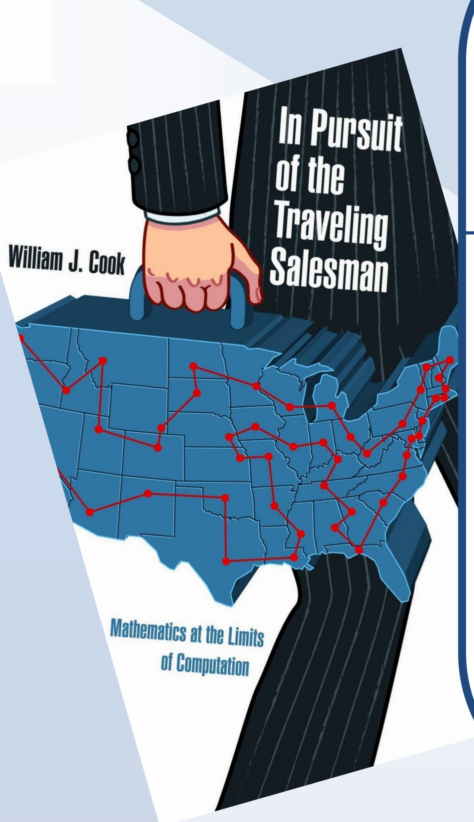

A Constant-Factor Approximation Algorithm for the Asymmetric Traveling Salesman Problem

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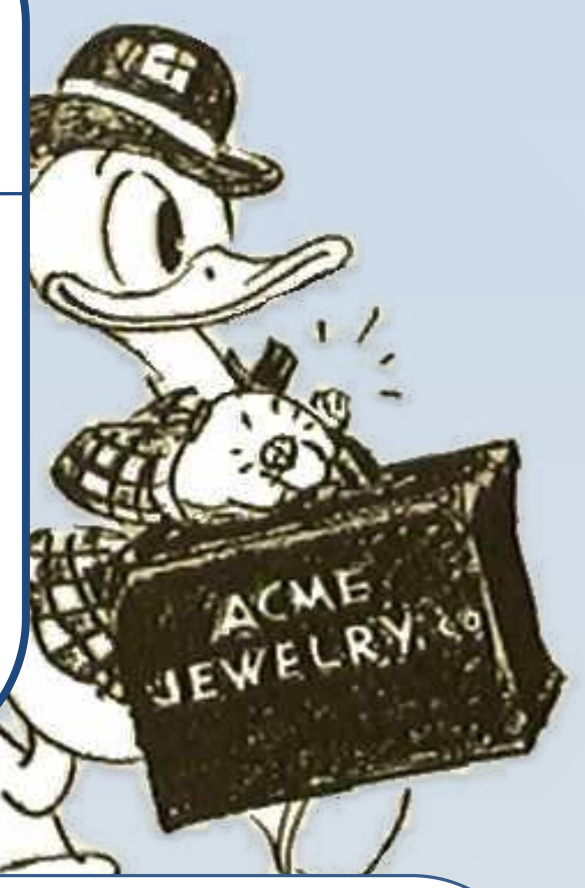
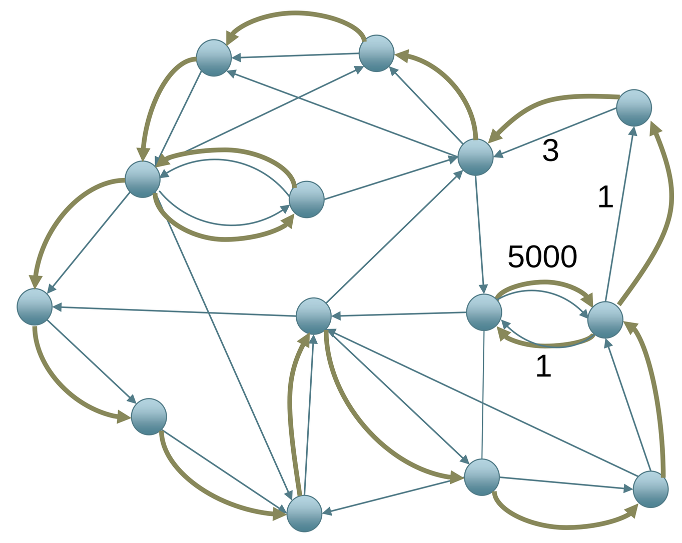


Asymmetric Traveling Salesman Problem

Input: edge-weighted directed graph $G = (V, E, w)$
Output: min-weight tour that visits each vertex at least once

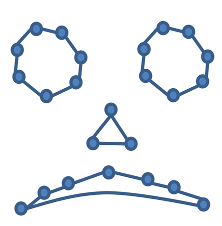


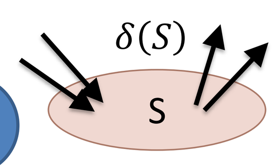
Symmetric TSP: 1.5-apx [Christofides'76]

What is the best possible apx ratio for Asymmetric TSP?

Held-Karp LP relaxation

Variables: x_{uv} = #times we traverse edge (u, v)
Minimize: $\sum_{uv \in E} w(u,v)\, x_{uv}$
Subject to: $x(\delta^+(v)) = x(\delta^-(v))$ for all $v \in V$ (indegree = outdegree: Eulerian)
$x(\delta(S)) \geq 2$ for all $S \subset V$ (eliminate subtours: connected)
$x \geq 0$





What is the integrality gap?

Previous work

Two main approaches:
- Add Eulerian graphs until connected
- Start with spanning tree, then make it Eulerian

- $\log_2 n$-apx via repeated cycle covers [Frieze, Galbiati, Maffioli'82]
- $0.99 \log_2 n$-apx [Bläser'03]
- $0.84 \log_2 n$-apx [Kaplan, Lewenstein, Shafrir, Sviridenko'05]
- $0.67 \log_2 n$-apx [Feige, Singh'07]
- $O(\log n / \log\log n)$-apx via thin trees [Asadpour, Goemans, Mądry, Oveis Gharan, Saberi'10]
- $O(1)$-apx for planar & bounded-genus graphs [Oveis Gharan, Saberi'11]
- Integrality gap $\leq \mathrm{poly}(\log\log n)$ via generalization of Kadison-Singer [Anari, Oveis Gharan'14]

Hardness of approximation:

$1 + \frac{1}{74}$ — $O(\log n / \log\log n)$

[Papadimitriou, Vempala'00, Karpinski, Lampis, Schmied'13]

Integrality gap:

2 — $\mathrm{poly}(\log\log n)$

[Charikar, Goemans, Karloff'02]

Local-Connectivity ATSP

[Svensson'15]:
- Defined new, easier problem called Local-Connectivity ATSP
- Reduced $O(1)$-apx of ATSP to Local-Connectivity ATSP
- Solved Local-Connectivity ATSP for unweighted graphs (easy part)

Thus: $O(1)$-apx of ATSP for unweighted graphs

Then we ['16] solved Local-Connectivity ATSP for graphs with two different edge weights. But could not generalize even to *three* weights…

Our result

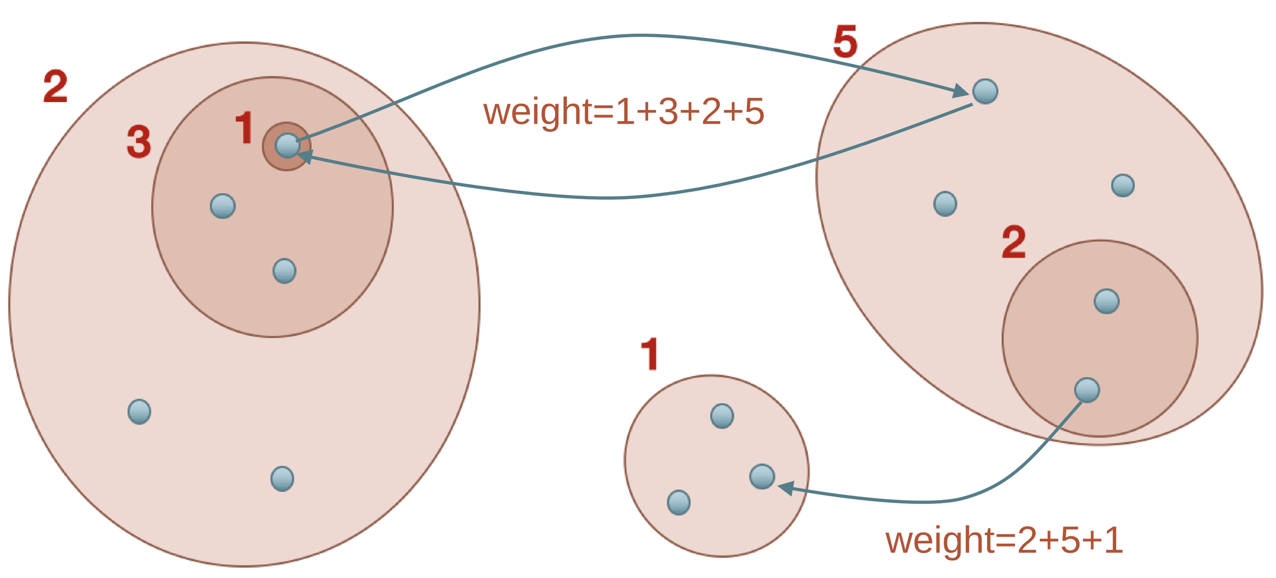
Theorem: $O(1)$-approximation algorithm for ATSP (with respect to Held-Karp relaxation)

Instead of solving Local-Connectivity ATSP directly for general instances, first simplify the ATSP instance via a **series of reductions** and solve Local-Connectivity ATSP for special, structured instances!

our apx ratio ≈ 5500

Make the instance *laminarly-weighted*

Solve dual, uncross, use complementary slackness, and rewrite objective function to get the following structure:



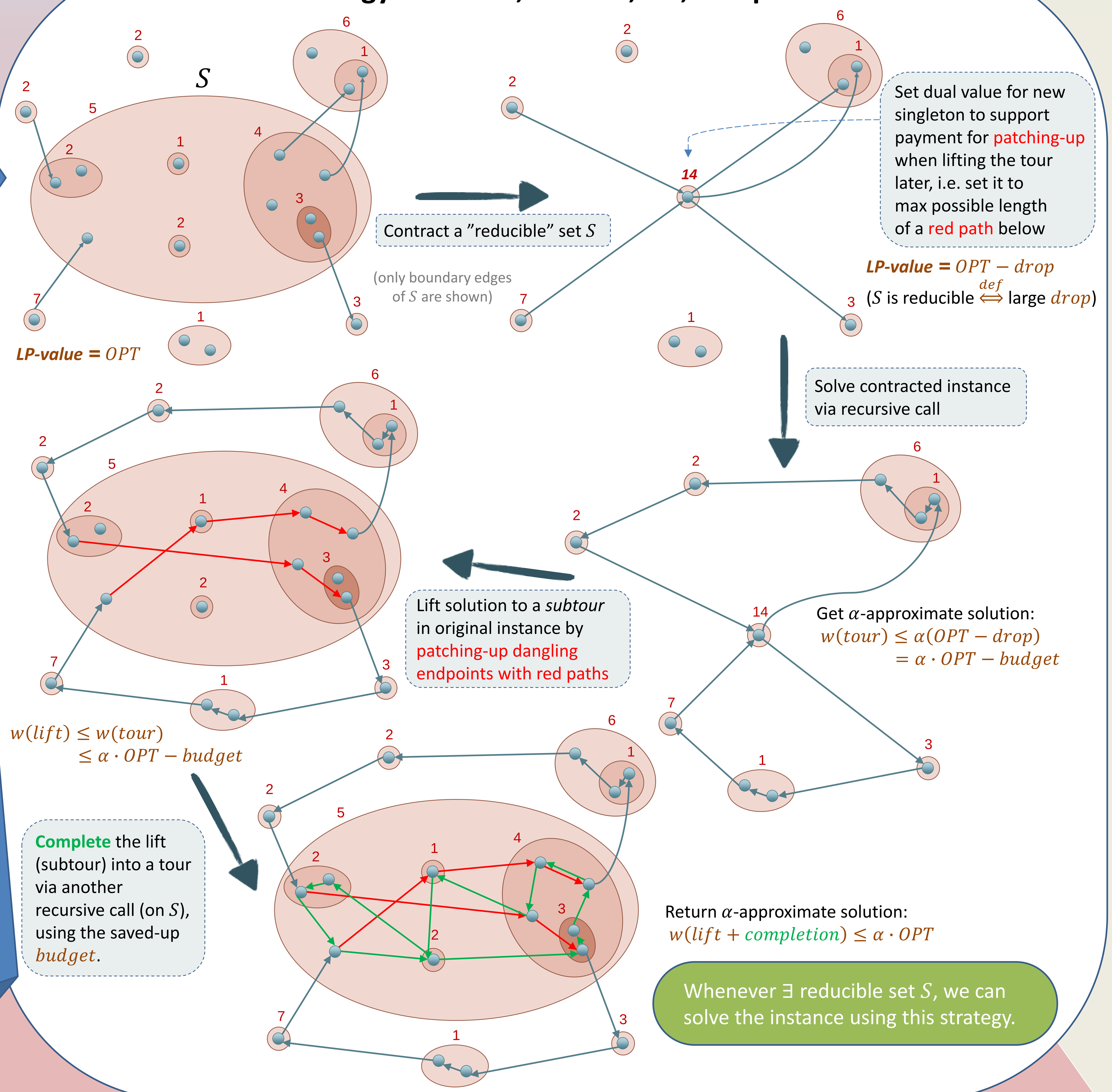
- **Laminar** family $\mathcal{L}$ of cuts with dual values $y_S > 0$
- Each cut is **tight**, i.e. LP says we should cross it once:
$x(\delta^+(S)) = x(\delta^-(S)) = 1$
- **Weight of each edge = total value of sets it crosses:**
$$w(e) = \sum_{S \in \mathcal{L}:\, e \in \delta(S)} y_S$$
(weights induced by $\mathcal{L}$ and y)
- Primal LP value = dual LP value:
$$\sum_{e \in E} w(e)\, x_e = 2 \cdot \sum_{S \in \mathcal{L}} y_S$$

Irreducible = close to Hamiltonian

S being reducible meant a large *drop* = ($\sum$ dual values of contracted sets) − (max length of a red path). But if *drop* is small, then longest red path visits almost all of the contracted sets inside S ⇒ S is close to Hamiltonian!

If all sets are close to Hamiltonian, then we have enough structure to solve Local-Connectivity ATSP using previous work, circulations, … … …

Strategy: contract, recurse, lift, complete



Whenever ∃ reducible set S, we can solve the instance using this strategy.

Future work

- Our apx ratio is not close to 2. At all. Need new ideas to get much better approximation algorithms for ATSP
- Bottleneck ATSP problem: find tour (visit each vertex *exactly* once) with minimum max edge-weight. Beat $O(\log n / \log\log n)$-apx
- Thin tree conjecture: find tree T such that for every $S \subset V$: $|\delta(S) \cap T| \leq O(1)\, x(\delta(S))$ (would imply $O(1)$-apx for ATSP and for Bottleneck ATSP)
- Node-weighted symmetric TSP, i.e. $w(u,v) = f(u) + f(v)$: beat 1.5-apx