Matching is in quasi-$\mathcal{NC}$

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Theory of Computation Lab

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

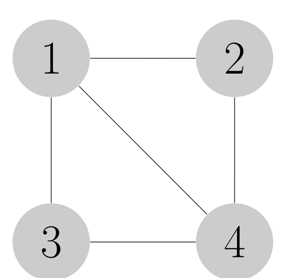
Perfect Matching

- **Perfect matching problem:** given graph G, determine if it has a perfect matching
- We can also: find a perfect matching, find a maximum matching, find min-weight perfect matching (if weights are small)
- Basic problem in graph theory and algorithms
- Polynomial-time deterministic algorithm known since the '60s

Parallel complexity

- **Class $\mathcal{NC}$:** polylog n time, poly(n) processors
- Completely parallelizable problems
- Perfect matching is in Randomized $\mathcal{NC}$
- **Do we need the randomness?** (is perfect matching in $\mathcal{NC}$?)
  - Lots of interest
  - Known for special graph classes: strongly chordal, planar bipartite, graphs with small number of perfect matchings, regular bipartite, P_4-tidy, dense, convex bipartite, claw-free, incomparability graphs...

Matrix approach



$$T(G) = \begin{pmatrix} 0 & X_{12} & X_{13} & X_{14} \\ -X_{12} & 0 & 0 & X_{24} \\ -X_{13} & 0 & 0 & X_{34} \\ -X_{14} & -X_{24} & -X_{34} & 0 \end{pmatrix}$$

- Build Tutte's matrix $T(G)$
- **Tutte's Theorem**: $\det T(G) \neq 0 \iff G$ has a perfect matching

Randomized $\mathcal{NC}$ algorithm

Due to Mulmuley, Vazirani and Vazirani (1987)

- Introduce a weight function!

$w : E \to \mathbb{Z}_+$ is **isolating** if there is unique perfect matching M with minimum $w(M)$

- In $T(G)$, substitute $X_{uv} := 2^{w(u,v)}$
- If w is isolating, then Tutte's Theorem still holds
- **Isolation Lemma**: assign polynomial weights randomly in $\{1, 2, ..., n^2\}$, then w isolating w.h.p.!

Algorithm

- Sample w (the only random component)
- Compute determinant (possible in $\mathcal{NC}$)
- Answer YES iff it is nonzero

Derandomize the Isolation Lemma!

- **Challenge**: deterministically get small set of weight functions (to be checked in parallel)
- **We prove**: can construct $n^{O(\log^2 n)}$ weight functions, with weights bounded by $n^{O(\log^2 n)}$, such that for any graph on n vertices, one of them is isolating
- Can even do it without looking at the graph
- **Implies: matching is in quasi-$\mathcal{NC}$** ($n^{\text{polylog}\, n}$ processors, polylog n time)
- Generalizes the approach by Fenner, Gurjar and Thierauf (2015) for bipartite graphs
- First step to derandomizing Polynomial Identity Testing?

The framework

w_1	$0\dots0$	w_2	$0\dots0$	$\cdots$	$w_{\ell-1}$	$0\dots0$	w_ℓ

- We concatenate multiple weight functions w_i, each from a small and simple set $\mathcal{W}$
- As we add new functions, think about set of min-weight matchings
- Begin from zero weight function; all perfect matchings are min-weight (their set $=: F_0$)
- Get decreasing sequence of sets of matchings

$$F_0 \supseteq F_1 \supseteq F_2 \supseteq ... \supseteq F_\ell$$
$$F_i = \text{argmin}\{\langle w_i, x\rangle : x \in F_{i-1}\}$$

- The concatenation is isolating if $|F_\ell| = 1$
- Just check all $n^{O(\log^2 n)}$ weight functions of this form :)

Main claim

Some $w_i \in \mathcal{W}$ will make F_i **two times smaller** than F_{i-1} (in some sense)

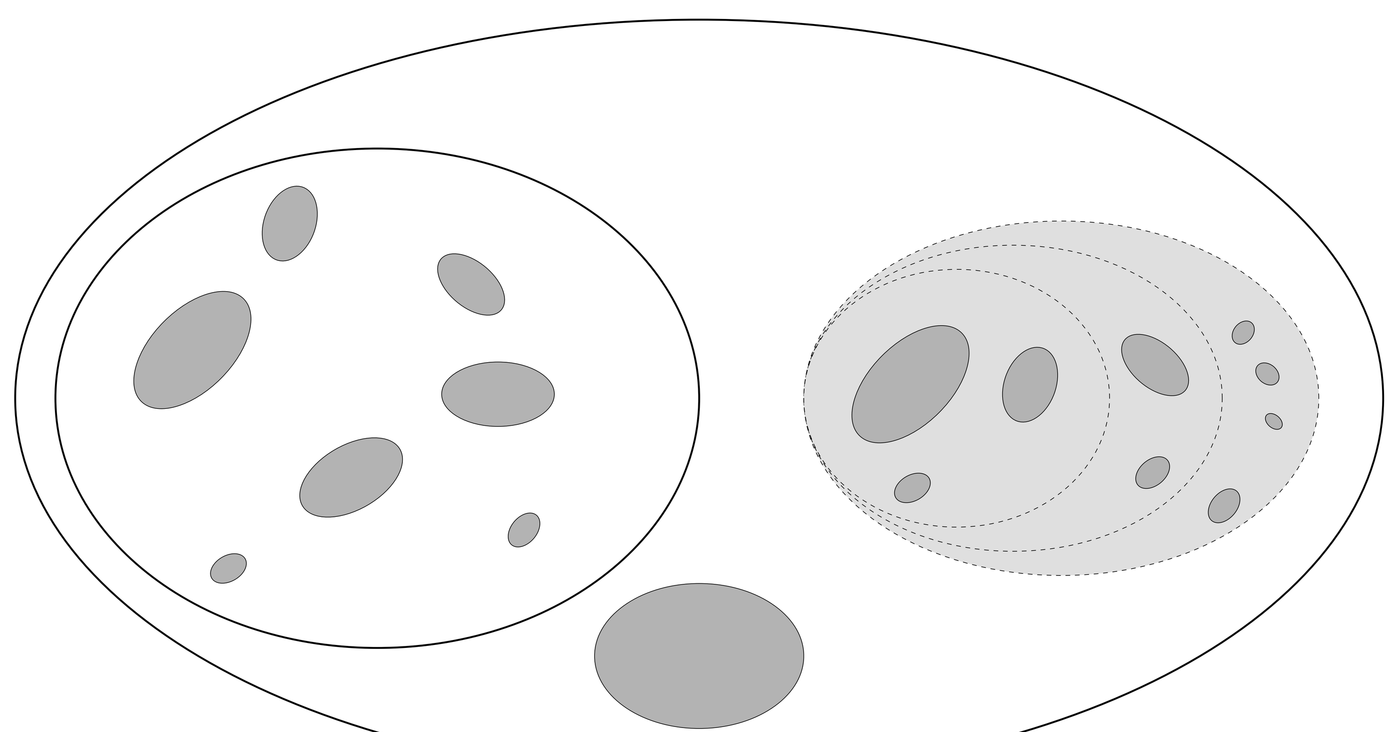
- In bipartite case, look at length of shortest cycle in the support of F_i – it doubles!
  - Short "alternating" cycles are being removed
  - One function from $\mathcal{W}$ can remove n^4 many
  - No cycles of length $\leq 2^i \implies$ only n^4 cycles of length $\leq 2^{i+1}$
- In general case, progress measure more involved

Polyhedral perspective

- Consider *convex hull* of min-weight perfect matchings
- It is a face of the perfect matching polytope
- Can be described by:
  - Subset of edges
  - **Laminar** family of tight odd-cuts

$$x(\delta(v)) = 1 \quad \text{for } v \in V$$
$$x(\delta(S)) \geq 1 \quad \text{for odd } S \subseteq V$$
$$x_e \geq 0 \quad \text{for } e \in E$$

- Use this family to define measure of progress
- Divide-and-conquer argument to isolate matchings in larger and larger parts of the graph



Still waiting for someone to:

- Go down to $\mathcal{NC}$ (even for bipartite graphs)
- *Find* perfect matching in a *planar* graph in $\mathcal{NC}$ (we can detect and even count)
- Exact matching problem: some edges are red; find perfect matching with exactly k red edges. We know a Randomized $\mathcal{NC}$ algorithm but not one in $\mathcal{P}$!
- Derandomize the Isolation Lemma for other polytopes (e.g. totally unimodular)?